Some Properties of rw -Sets and rw -Continuous Functions¹

Philip Lester P. Benjamin and Helen M. Rara

Department of Mathematics
Mindanao State University - Iligan Institute of Technology
Tibanga, Iligan city, Philippines

Copyright © 2014 Philip Lester P. Benjamin and Helen M. Rara. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, the concept of regular w -closed (rw -closed) sets in topological spaces introduced in [1] is further studied. It also investigates related concepts such as rw -interior and rw -closure of a set, and rw -continuous.

Mathematics Subject Classification: 54A05

Keywords: regular open sets, rw -sets, rw -functions

1 Introduction

In 1937, Stone [6] introduced and investigated the regular open sets. These sets are contained in the family of open sets since a set is regular open if it is equal to the interior of its closure. In 1978, Cameron [2] also introduced and investigated the concept of a regular semiopen set. A set A is regular semiopen if there is a regular open set U such that $U \subseteq A \subseteq \overline{U}$. In 2007, a new class of sets called regular w -closed sets (rw -closed sets) was introduced by Benchalli and Wali [1]. A set B is rw -closed if $\overline{B} \subseteq U$ whenever $B \subseteq U$ for any regular semiopen set U . They proved that this new class of sets is properly placed

¹This research is funded by the Department of Science and Technology-Philippine Council for Advanced Science and Technology Research and Development (DOST-PCASTRD).

in between the class of w -closed sets [5] and the class of regular generalized closed sets [4].

In this paper, the concepts of rw -closed and rw -open sets (complement of rw -closed set) are further investigated. Also, the study of related functions involving rw -closed and rw -open sets are characterized.

Throughout this paper, space (X, T) (or simply X) always means a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X , $\overline{A}$, $\text{int}(A)$, and $C(A)$ denote the closure of A , interior of A , and complement of A in X , respectively.

2 Preliminaries

Definition 2.1 [1] A subset A of a space X is called

- (i) *regular open* if $\text{int}(\overline{A}) = A$ and it is *regular closed* if $\overline{\text{int}(A)} = A$.
- (ii) *regular semiopen* if there exists a regular open set U such that $U \subseteq A \subseteq \overline{U}$.
- (iii) *regular w -closed set* (briefly, *rw -closed*) if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is regular semiopen in X . The complement of any rw -closed set is called *rw -open set*.

Definition 2.2 [3] The intersection of all the rw -closed sets of X containing A is called the *rw -closure* of A , denoted by $rw(\overline{A})$.

Definition 2.3 [3] The union of all the rw -open sets of a space X contained in A is called the *rw -interior* of A , denoted by $rw\text{-int}(A)$.

Definition 2.4 [1] A function $f : X \rightarrow Y$ is called

- (i) *rw -open* if the image $f(A)$ is rw -open in Y for each open set A in X .
- (ii) *rw -closed* if the image $f(A)$ is rw -closed for each closed set A in X .
- (iii) *rw -continuous* if for every open subset U of Y , $f^{-1}(U)$ is rw -open in X .

Theorem 2.5 [1] *Every closed set is rw -closed.*

3 rw -interior and rw -closure of a Set

Theorem 3.1 *Let (X, T) be a topological space and $A, B \subseteq X$. Then*

- (a) *If A is open, then A is rw -open.*
- (b) *If A is rw -open, then $A = rw\text{-int}(A)$.*

- (c) $\text{int}(A) \subseteq \text{rw-int}(A)$.
- (d) If $A \subseteq B$, then $\text{rw-int}(A) \subseteq \text{rw-int}(B)$.
- (e) If A and B are both rw-open, then $A \cap B$ is rw-open.

Remark 3.2 The converses of Theorem 3.1 (a) and (b) are not true.

Remark 3.3 Let (X, T) be a topological space and $A, B \subseteq X$. If A and B are both rw-open, then $A \cup B$ need not be rw-open. Thus, the family of all the rw-open subsets of X is not a topology in X .

Theorem 3.4 A is rw-open in X if and only if for every regular semiopen set U in X with $A \cup U = X$, $\text{int}(A) \cup U = X$.

Proof: ($\Rightarrow$) Let A be an rw-open set in X and let U be a regular semiopen with $A \cup U = X$. Then $C(A) \cap C(U) = \emptyset$ implying that $C(A) \subseteq U$. Since $C(A)$ is rw-closed, $\overline{C(A)} \subseteq U$. Hence $C(U) \subseteq C(\overline{C(A)})$. But $C(\overline{C(A)}) = \text{int}(A)$. Thus $C(U) \subseteq \text{int}(A)$. Therefore, $\text{int}(A) \cup U = X$.

($\Leftarrow$) Let U be a regular semiopen set such that $C(A) \subseteq U$. Then $C(A) \cap C(U) = \emptyset$ implying that $A \cup U = X$. By hypothesis, $\text{int}(A) \cup U = X$ implies that $C(U) \subseteq \text{int}(A) = C(\overline{C(A)})$ so that $\overline{C(A)} \subseteq U$. Thus $C(A)$ is rw-closed. Consequently, A is rw-open. $\square$

Theorem 3.5 Let (X, T) be a topological space and $A, B \subseteq X$. Then

- (a) $x \in \text{rw}-(\overline{A})$ if and only if for every rw-open set O with $x \in O$, $O \cap A \neq \emptyset$.
- (b) For any set A , $\text{rw}-(\overline{A}) \subseteq \overline{\text{rw}-(\overline{A})}$.
- (c) If A is rw-closed, then $A = \text{rw}-(\overline{A}) = \overline{\text{rw}-(\overline{A})}$.
- (d) $\text{rw}-(\overline{A \cup B}) = \text{rw}-(\overline{A}) \cup \text{rw}-(\overline{B})$.
- (e) $\text{rw}-(\overline{A}) \subseteq \overline{A}$.
- (f) If A and B are subsets of X with $A \subseteq B$, then $\text{rw}-(\overline{A}) \subseteq \text{rw}-(\overline{B})$.

4 *rw*-continuous Functions

Theorem 4.1 *Every continuous function is *rw*-continuous.*

Proof: Let X and Y be topological spaces and let $f : X \rightarrow Y$ be a function. Suppose that A is any open set in Y . Since f is continuous, $f^{-1}(A)$ is open in X . By Theorem 3.1(a), $f^{-1}(A)$ is *rw*-open. Thus, f is *rw*-continuous. $\square$

Theorem 4.2 *If $f : X \rightarrow Y$ is *rw*-continuous and $g : Y \rightarrow Z$ is continuous, then $g \circ f : X \rightarrow Z$ is *rw*-continuous.*

Proof: Let U be open in Z . Then $g^{-1}(U)$ is open since g is continuous. Thus, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is *rw*-open since f is *rw*-continuous. Therefore, $g \circ f$ is *rw*-continuous. $\square$

Remark 4.3 *The composition of two *rw*-continuous functions need not be *rw*-continuous.*

Theorem 4.4 *Let X and Y be topological spaces and $f : X \rightarrow Y$. Then f is *rw*-continuous if and only if the inverse image of each closed set in Y is *rw*-closed in X .*

Proof: Let f be *rw*-continuous and let U be any closed set in Y . Then $Y \setminus U$ is open. Since f is *rw*-continuous, $f^{-1}(Y \setminus U)$ is *rw*-open. Now,

$$f^{-1}(Y \setminus U) = f^{-1}(Y) \setminus f^{-1}(U) = X \setminus f^{-1}(U).$$

Hence, $f^{-1}(U)$ is *rw*-closed in X .

Conversely, let U be open in Y . Then $Y \setminus U$ is closed. By assumption, $f^{-1}(Y \setminus U)$ is *rw*-closed in X . Now,

$$f^{-1}(Y \setminus U) = f^{-1}(Y) \setminus f^{-1}(U) = X \setminus f^{-1}(U).$$

Hence, $f^{-1}(U)$ is *rw*-open. Therefore, f is *rw*-continuous. $\square$

Theorem 4.5 *If $f : X \rightarrow Y$ is *rw*-continuous, then $f(rw-(\overline{A})) \subseteq \overline{f(A)}$ for every $A \subseteq X$.*

Proof: Let $A \subseteq X$ and let $x \in rw-(\overline{A})$. Suppose further that U is an open set in Y with $f(x) \in U$. Since f is *rw*-continuous, $f^{-1}(U)$ is *rw*-open in X with $x \in f^{-1}(U)$. Hence, by Theorem 3.5(a), $f^{-1}(U) \cap A \neq \emptyset$. It follows that

$$\emptyset \neq f(f^{-1}(U) \cap A) \subseteq f(f^{-1}(U)) \cap f(A) \subseteq U \cap f(A).$$

Thus, $U \cap f(A) \neq \emptyset$. Hence, $f(x) \in \overline{f(A)}$. $\square$

Theorem 4.6 *If $f : X \rightarrow Y$ is *rw*-continuous, then $rw-(\overline{f^{-1}(B)}) \subseteq f^{-1}(\overline{B})$ for every $B \subseteq Y$.*

Proof: Let $f : X \rightarrow Y$ be *rw*-continuous. Suppose that $B \subseteq Y$ and $A = f^{-1}(B)$. Then by Theorem 4.5, $f(rw-(\overline{f^{-1}(B)})) \subseteq \overline{f(f^{-1}(B))} \subseteq \overline{B}$. Thus, $rw-(\overline{f^{-1}(B)}) \subseteq f^{-1}(\overline{B})$. $\square$

Definition 4.7 A function $f : X \rightarrow Y$ is called *regular strongly continuous* (briefly *rs*-continuous) if the inverse image of every *rw*-open set in Y is open in X , that is, $f^{-1}(A)$ is open in X for all *rw*-open sets A in Y .

Remark 4.8 *Every *rs*-continuous function is *rw*-continuous.*

Theorem 4.9 *$f : X \rightarrow Y$ is *rs*-continuous if and only if $f^{-1}(A)$ is closed for every *rw*-closed set A in Y .*

Proof: ($\Rightarrow$) Let f be *rs*-continuous and let A be *rw*-closed in Y . Then $C(A)$ is *rw*-open in Y . Thus, $f^{-1}(C(A))$ is open since f is *rs*-continuous. But $f^{-1}(C(A)) = C(f^{-1}(A))$. Hence, $f^{-1}(A)$ is closed.

($\Leftarrow$) Let O be *rw*-open in Y . Then $C(O)$ is *rw*-closed. By assumption, $f^{-1}(C(O))$ is closed. Thus, $f^{-1}(C(O)) = C(f^{-1}(O))$ is closed. Therefore, $f^{-1}(O)$ is open implying that f is *rs*-continuous. $\square$

References

- [1] S.S Benchalli and R.S Wali. "On *rw*-closed sets in Topological Spaces". *Bulletin of the Malaysian Mathematical Sciences Society*.(2)30(2)(2007),99-110.
- [2] D.E Cameron, Properties of S-closed spaces, *Proc. Amer. Math. Soc.*, 72(1978), 581-586.
- [3] V. Chandrasekar, M. Saraswathi and A. Vadivel, "Slightly *rw*-continuous Functions". *Int. J. Contemp. Math. Sciences* 5,no. 40, pp.1985-1994. 2010.
- [4] N. Palaniappan and K.C Rao, Regular generalized closed sets, *Kyungpook Math J.*,33(1993),211-219.
- [5] A. Pushpalatha, *Studies on generalizations of mappings in topological spaces*, Ph.D Thesis, Bharathiar University Coimbatore, 2000.
- [6] M. Stone, Application of the theory of Boolean rings to general topology, *Trans. Amer. Math. Soc.* 41(1937),374-481.

Receied: March 15, 2014