Some Properties of $rw$-Sets and $rw$-Continuous Functions\(^1\)

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Abstract

In this paper, the concept of regular $w$-closed ($rw$-closed) sets in topological spaces introduced in [1] is further studied. It also investigates related concepts such as $rw$-interior and $rw$-closure of a set, and $rw$-continuous.

Mathematics Subject Classification: 54A05

Keywords: regular open sets, $rw$-sets, $rw$-functions

1 Introduction

In 1937, Stone [6] introduced and investigated the regular open sets. These sets are contained in the family of open sets since a set is regular open if it is equal to the interior of its closure. In 1978, Cameron [2] also introduced and investigated the concept of a regular semiopen set. A set $A$ is regular semiopen if there is a regular open set $U$ such that $U \subseteq A \subseteq U$. In 2007, a new class of sets called regular $w$-closed sets ($rw$-closed sets) was introduced by Benchalli and Wali [1]. A set $B$ is $rw$-closed if $B \subseteq U$ whenever $B \subseteq U$ for any regular semiopen set $U$. They proved that this new class of sets is properly placed

\(^1\)This research is funded by the Department of Science and Technology-Philippine Council for Advanced Science and Technology Research and Development (DOST-PCASTRD).
in between the class of \( w \)-closed sets [5] and the class of regular generalized closed sets [4].

In this paper, the concepts of \( rw \)-closed and \( rw \)-open sets (complement of \( rw \)-closed set) are further investigated. Also, the study of related functions involving \( rw \)-closed and \( rw \)-open sets are characterized.

Throughout this paper, space \((X, T)\) (or simply \(X\)) always means a topological space on which no separation axioms are assumed unless explicitly stated. For a subset \(A\) of a space \(X\), \(\overline{A}\), \(\text{int}(A)\), and \(C(A)\) denote the closure of \(A\), interior of \(A\), and complement of \(A\) in \(X\), respectively.

## 2 Preliminaries

**Definition 2.1** [1] A subset \(A\) of a space \(X\) is called
(i) regular open if \(\text{int}(\overline{A}) = A\) and it is regular closed if \(\text{int}(A) = A\).
(ii) regular semiopen if there exists a regular open set \(U\) such that \(U \subseteq A \subseteq \overline{U}\).
(iii) regular \(w\)-closed set (briefly, \(rw\)-closed) if \(A \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular semiopen in \(X\). The complement of any \(rw\)-closed set is called \(rw\)-open set.

**Definition 2.2** [3] The intersection of all the \(rw\)-closed sets of \(X\) containing \(A\) is called the \(rw\)-closure of \(A\), denoted by \(\text{rw-}(A)\).

**Definition 2.3** [3] The union of all the \(rw\)-open sets of a space \(X\) contained in \(A\) is called the \(rw\)-interior of \(A\), denoted by \(\text{rw-int}(A)\).

**Definition 2.4** [1] A function \(f : X \rightarrow Y\) is called
(i) \(rw\)-open if the image \(f(A)\) is \(rw\)-open in \(Y\) for each open set \(A\) in \(X\).
(ii) \(rw\)-closed if the image \(f(A)\) is \(rw\)-closed for each closed set \(A\) in \(X\).
(iii) \(rw\)-continuous if for every open subset \(U\) of \(Y\), \(f^{-1}(U)\) is \(rw\)-open in \(X\).

**Theorem 2.5** [1] Every closed set is \(rw\)-closed.

## 3 \( rw \)-interior and \( rw \)-closure of a Set

**Theorem 3.1** Let \((X, T)\) be a topological space and \(A, B \subseteq X\). Then

(a) If \(A\) is open, then \(A\) is \(rw\)-open.

(b) If \(A\) is \(rw\)-open, then \(A = \text{rw-int}(A)\).
(c) int(A) ⊆ rw-int(A).

(d) If A ⊆ B, then rw-int(A) ⊆ rw-int(B).

(e) If A and B are both rw-open, then A ∩ B is rw-open.

Remark 3.2 The converses of Theorem 3.1 (a) and (b) are not true.

Remark 3.3 Let (X, T) be a topological space and A, B ⊆ X. If A and B are both rw-open, then A ∪ B need not be rw-open. Thus, the family of all the rw-open subsets of X is not a topology in X.

Theorem 3.4 A is rw-open in X if and only if for every regular semiopen set U in X with A ∪ U = X, int(A) ∪ U = X.

Proof: (⇒) Let A be an rw-open set in X and let U be a regular semiopen with A ∪ U = X. Then C(A) ∩ C(U) = ∅ implying that C(A) ⊆ U. Since C(A) is rw-closed, C(A) ⊆ U. Hence C(U) ⊆ C(C(A)). But C(C(A)) = int(A). Thus C(U) ⊆ int(A). Therefore, int(A) ∪ U = X.

(⇐) Let U be a regular semiopen set such that C(A) ⊆ U. Then C(A) ∩ C(U) = ∅ implying that A ∪ U = X. By hypothesis, int(A) ∪ U = X implies that C(U) ⊆ int(A) = C(C(A)) so that C(A) ⊆ U. Thus C(A) is rw-closed. Consequently, A is rw-open. □

Theorem 3.5 Let (X, T) be a topological space and A, B ⊆ X. Then

(a) x ∈ rw-(A) if and only if for every rw-open set O with x ∈ O, O ∩ A ≠ ∅.

(b) For any set A, rw-(A) ⊆ rw-(rw-(A)).

(c) If A is rw-closed, then A = rw-(A) = rw-(rw-(A)).

(d) rw-(A ∪ B) = rw-(A) ∪ rw-(B).

(e) rw-(A) ⊆ A.

(f) If A and B are subsets of X with A ⊆ B, then rw-(A) ⊆ rw-(B).
4 \textit{rw}-continuous Functions

\textbf{Theorem 4.1} Every continuous function is \textit{rw}-continuous.

\textit{Proof}: Let \(X\) and \(Y\) be topological spaces and let \(f : X \rightarrow Y\) be a function. Suppose that \(A\) is any open set in \(Y\). Since \(f\) is continuous, \(f^{-1}(A)\) is open in \(X\). By Theorem 3.1(a), \(f^{-1}(A)\) is \(\textit{rw}\)-open. Thus, \(f\) is \(\textit{rw}\)-continuous. \(\square\)

\textbf{Theorem 4.2} If \(f : X \rightarrow Y\) is \(\textit{rw}\)-continuous and \(g : Y \rightarrow Z\) is continuous, then \(g \circ f : X \rightarrow Z\) is \(\textit{rw}\)-continuous.

\textit{Proof}: Let \(U\) be open in \(Z\). Then \(g^{-1}(U)\) is open since \(g\) is continuous. Thus, \(f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)\) is \(\textit{rw}\)-open since \(f\) is \(\textit{rw}\)-continuous. Therefore, \(g \circ f\) is \(\textit{rw}\)-continuous. \(\square\)

\textbf{Remark 4.3} The composition of two \(\textit{rw}\)-continuous functions need not be \(\textit{rw}\)-continuous.

\textbf{Theorem 4.4} Let \(X\) and \(Y\) be topological spaces and \(f : X \rightarrow Y\). Then \(f\) is \(\textit{rw}\)-continuous if and only if the inverse image of each closed set in \(Y\) is \(\textit{rw}\)-closed in \(X\).

\textit{Proof}: Let \(f\) be \(\textit{rw}\)-continuous and let \(U\) be any closed set in \(Y\). Then \(Y \setminus U\) is open. Since \(f\) is \(\textit{rw}\)-continuous, \(f^{-1}(Y \setminus U)\) is \(\textit{rw}\)-open. Now,

\[ f^{-1}(Y \setminus U) = f^{-1}(Y) \setminus f^{-1}(U) = X \setminus f^{-1}(U). \]

Hence, \(f^{-1}(U)\) is \(\textit{rw}\)-closed in \(X\).

Conversely, let \(U\) be open in \(Y\). Then \(Y \setminus U\) is closed. By assumption, \(f^{-1}(Y \setminus U)\) is \(\textit{rw}\)-closed in \(X\). Now,

\[ f^{-1}(Y \setminus U) = f^{-1}(Y) \setminus f^{-1}(U) = X \setminus f^{-1}(U). \]

Hence, \(f^{-1}(U)\) is \(\textit{rw}\)-open. Therefore, \(f\) is \(\textit{rw}\)-continuous. \(\square\)

\textbf{Theorem 4.5} If \(f : X \rightarrow Y\) is \(\textit{rw}\)-continuous, then \(f(\textit{rw}-(A)) \subseteq \overline{f(A)}\) for every \(A \subseteq X\).

\textit{Proof}: Let \(A \subseteq X\) and let \(x \in \textit{rw}-(A)\). Suppose further that \(U\) is an open set in \(Y\) with \(f(x) \in U\). Since \(f\) is \(\textit{rw}\)-continuous, \(f^{-1}(U)\) is \(\textit{rw}\)-open in \(X\) with \(x \in f^{-1}(U)\). Hence, by Theorem 3.5(a), \(f^{-1}(U) \cap A \neq \emptyset\). It follows that

\[ \emptyset \neq f(f^{-1}(U) \cap A) \subseteq f(f^{-1}(U)) \cap f(A) \subseteq U \cap f(A). \]

Thus, \(U \cap f(A) \neq \emptyset\). Hence, \(f(x) \in \overline{f(A)}\). \(\square\)
Theorem 4.6 If \( f : X \rightarrow Y \) is \( rw \)-continuous, then \( rw(f^{-1}(B)) \subseteq f^{-1}(B) \) for every \( B \subseteq Y \).

Proof: Let \( f : X \rightarrow Y \) be \( rw \)-continuous. Suppose that \( B \subseteq Y \) and \( A = f^{-1}(B) \). Then by Theorem 4.5, \( f(rw(f^{-1}(B))) \subseteq f(f^{-1}(B)) \subseteq B \). Thus, \( rw(f^{-1}(B)) \subseteq f^{-1}(B) \). □

Definition 4.7 A function \( f : X \rightarrow Y \) is called regular strongly continuous (briefly \( rs \)-continuous) if the inverse image of every \( rw \)-open set in \( Y \) is open in \( X \), that is, \( f^{-1}(A) \) is open in \( X \) for all \( rw \)-open sets \( A \) in \( Y \).

Remark 4.8 Every \( rs \)-continuous function is \( rw \)-continuous.

Theorem 4.9 \( f : X \rightarrow Y \) is \( rs \)-continuous if and only if \( f^{-1}(A) \) is closed for every \( rw \)-closed set \( A \) in \( X \).

Proof: (⇒) Let \( f \) be \( rs \)-continuous and let \( A \) be \( rw \)-closed in \( Y \). Then \( C(A) \) is \( rw \)-open in \( Y \). Thus, \( f^{-1}(C(A)) \) is open since \( f \) is \( rs \)-continuous. But \( f^{-1}(C(A)) = C(f^{-1}(A)) \). Hence, \( f^{-1}(A) \) is closed.

(⇐) Let \( O \) be \( rw \)-open in \( Y \). Then \( C(O) \) is \( rw \)-closed. By assumption, \( f^{-1}(C(O)) \) is closed. Thus, \( f^{-1}(C(O)) = C(f^{-1}(O)) \) is closed. Therefore, \( f^{-1}(O) \) is open implying that \( f \) is \( rs \)-continuous. □

References


Received: March 15, 2014