Noor Iterative Processes for Multivalued Mappings
in Banach Spaces

D. P. Shukla, Vivek Tiwari and Ruchira Singh

Department of mathematics & computer Science
Govt. Model Science College, Rewa, (M.P.), India,486001

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Abstract

Let $K$ be a nonempty compact convex subset of a uniformly convex Banach space, and $T : K \to \mathcal{P}(K)$ a multivalued nonexpansive mapping. We prove that the sequences of Noor iterate converge to a fixed point of $T$. This generalizes former results proved by Banach convergence of Noor iterates for a multi-valued mapping with a fixed point. We also introduce both of the iterative processes in a new sense, and prove a convergence theorem of Noor iterates for a mapping defined on a noncompact domain.

Keywords: Multivalued mappings; Fixed points; Noor iterates; uniformly convex Banach space

1. Introduction

Let $K$ be a nonempty bounded closed convex subset of a Banach space $X$. A mapping $T : K \to K$ is said to be nonexpansive if

$$\|Tx - Tx\| \leq \|x - y\|, \quad \text{for all } x, y \in K$$
It has been shown that if $X$ is uniformly convex then every nonexpansive mapping $T : K \to K$ has a fixed point (see Browder [2], cf. also KirK [3]). In 1974, Ishikawa [4] introduced a new iteration procedure for approximating fixed point of pseudo-contractive compact mapping in Hilbert space as follows.

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T[\beta_n x_n + (1-\beta_n)Tx_n], \quad n \geq 0,$$

Where $\{ \alpha_n \}$ and $\{ \beta_n \}$ are sequence in $[0, 1]$ satisfying certain restrictions. Note that the normal Mann iteration procedure [5],

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0,$$

Where $\{ \alpha_n \}$ is a sequence in $[0, 1]$, is a special case of the Ishikawa one. For a comparison of the two iterative processes in the one-dimensional case, we refer the reader to Rhoades [6]. For more details and Literature on the convergence of Ishikawa and Mann iterates we refer to [7-14]. Recently, Sastry and Babu [1] introduced the analogs of Mann and Ishikawa iterates for nonexpansive mappings whose domain is a compact convex subset of a Hilbert space. In this paper, we generalize results of Sastry and Babu to uniformly convex Banach spaces. We also introduce both of the iteration processes in a new sense, and prove a convergence theorem of Mann iterates for a mapping defined on a noncompact domain.

2. Preliminaries

Let $X$ be a Banach space, $K$ be a nonempty, convex subset of $X$, and $T$ be a self map of $K$. Three most popular iteration procedures for obtaining fixed points of $T$, if they exist, we defined Noor iteration as follows:

Noor iteration [17], defined by

$$x_1 \in K, \ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n,$$

$$y_n = (1-\beta_n)x_n + \beta_n Tz_n,$$

$$z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \quad n \geq 1,$$
Let $X$ be a Banach space. A subset $K$ is call proximinal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x, k) = \text{dist}(x, K) = \inf \|x - y\| : y \in K.$$ 

It is well known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote by $P(K)$ the family of nonempty bounded proximinal subset of $K$. Let $H(., .)$ be the Hausdorff distance on $P(K)$, i.e.,

$$H(A, B) = \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\}, \quad A, B \in P(K),$$

Where $\text{dist}(a, B) = \inf \|a - b\| : b \in B$ is the distance from the point $a$ to the set $B$.

A multivalued mappings $T : K \rightarrow P(K)$ is said to be a nonexpansive if

$$H(Tx, Ty) \leq \|x - y\| \quad \text{for all } x, y \in K.$$

A point $x$ is called a fixed point of $T$ if $x \in Tx$. the existence of fixed points for multivalued nonexpansive mappings in uniformly convex Banach space was proved by Lim [15]. From now on, $X$ stands for a uniformly convex Banach space and $F(T)$ stands for the fixed point set of a mapping $T$.

**Definition 2.1 ([1])** Let $K$ be a nonempty convex subset of $X$, $T : K \rightarrow P(K)$ a multivalued mapping and fix $p \in F(T)$.

The sequence of Mann iterates is defined by $x_0 \in K$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \quad \alpha_n \in [0,1], n \geq 0,$$

Where $y_n \in Tx_n$ is such that $\|y_n - p\| = \text{dist}(p, Tx_n)$,

The sequence of Ishikawa iterates is defined by $x_0 \in K$,

$$y_n = (1 - \beta_n) x_n + \beta_n z_n, \quad \beta_n \in [0,1], n \geq 0$$
Where $z_n \in Tx_n$ is such that $\|z_n - p\| = \text{dist}(p, Tx_n)$, and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z_n^\prime, \quad \alpha_n \in [0,1]$$

Where $z_n^\prime \in Ty_n$ is such that $\|z_n^\prime - p\| = \text{dist}(p, Ty_n)$.

The sequence of Noor iterates is defined by $x_0 \in K$,

(C)

$$z_n = (1 - \gamma_n)x_n + \gamma_n z_n$$

Where $z_n \in Tx_n$ is such that $\|z_n - p\| = \text{dist}(p, Tx_n)$,

$$y_n = (1 - \beta_n)x_n + \beta_n z_n^\prime, \quad \beta_n \in [0,1], n \geq 0$$

Where $z_n^\prime \in Tz_n$ is such that $\|z_n^\prime - p\| = \text{dist}(p, Tz_n)$, and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z_n^\prime, \quad \alpha_n \in [0,1]$$

Where $z_n^\prime \in Ty_n$ is such that $\|z_n^\prime - p\| = \text{dist}(p, Ty_n)$.

**Lemma 2.2.** [1] Let $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences such that

(i) $0 \leq \alpha_n, \beta_n < 1$,

(ii) $\beta_n \to 0$ as $n \to \infty$ and

(iii) $\sum \alpha_n \beta_n = \infty$.

Let $\{\gamma_n\}$ be a nonnegative real sequence such that $\sum \alpha_n \beta_n (1 - \beta_n) \gamma_n$ is bounded. Then $\{\gamma_n\}$ has a subsequence which convergence to zero.

**Lemma 2.3.** Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences such that

(i) $0 \leq \alpha_n, \beta_n < 1, \gamma_n < 1$
Noor iterative processes for multivalued mappings  

\( \beta_n \to 0, \gamma_n \to 0 \) as \( n \to \infty \) and

\( \sum \alpha_n \beta_n \gamma_n = \infty. \)

Let \( \{\delta_n\} \) be a nonnegative real sequence such that \( \sum \beta_n \gamma_n (1 - \gamma_n) \delta_n \) is bounded. Then \( \{\delta_n\} \) has a subsequence which convergence to zero.

**Lemma 2.4.** Let \( X \) be a Banach space. Then \( X \) is uniformly convex if and only if for any given number \( \rho > 0, \)

The square norm \( \| \cdot \|^2 \) of \( X \) uniformly convex on \( B_\rho \), the closed ball centered at the origin with radius \( \rho \); namely, there exists a continuous strictly increasing function \( \varphi : [0, \infty) \to [0, \infty) \) with \( \varphi(0) = 0 \) such that

\[
\| \alpha x + (1 - \alpha) y \| \leq \alpha \| x \|^2 + (1 - \alpha) \| y \|^2 - \alpha (1 - \alpha) \varphi(\| x - y \|),
\]

For all \( x, y \in B_\rho, \alpha \in [0, 1]. \)

**3. Main results**

**Theorem 3.1.** Let \( K \) be a nonempty compact convex subset of a uniformly convex Banach space \( X \). Suppose that a nonexpansive map \( T : K \to P(K) \) has a fixed point \( p \). Let \( \{x_n\} \) be the sequence of Noor iterates defined by (C). Assume that

(i) \( 0 \leq \alpha_n, \beta_n < 1, \gamma_n < 1 \)

(ii) \( \beta_n \to 0, \gamma_n \to 0 \) and

(iii) \( \sum \alpha_n \beta_n \gamma_n = \infty. \) Then the sequence \( \{x_n\} \) convergence to a fixed point of \( T. \)

**Proof.** By using Lemma 2.4, we have
\[
\|x_{n+1} - p\|^2 = \|(1 - \alpha_n)x_n + \alpha_n z_n - p\|^2 \\
\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n \|z_n - p\|^2 - \alpha_n(1 - \alpha_n)\phi(\|x_n - z_n\|)
\]

\[
\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n H^2(Ty_n, Tp) - \alpha_n(1 - \alpha_n)\phi(\|x_n - z_n\|)
\]

\[
\leq (1 - \alpha_n)\|x_n - p\|^2 - \alpha_n(1 - \alpha_n)\phi(\|x_n - z_n\|)
\]

(2)

\[
\|y_n - p\|^2 = \|(1 - \beta_n)x_n + \beta_n z_n - p\|^2 \\
\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n \|z_n - p\|^2 - \beta_n(1 - \beta_n)\phi(\|x_n - z_n\|)
\]

\[
\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n H^2(Tz_n, Tp) - \beta_n(1 - \beta_n)\phi(\|x_n - z_n\|)
\]

\[
\leq (1 - \beta_n)\|x_n - p\|^2 - \beta_n(1 - \beta_n)\phi(\|x_n - z_n\|)
\]

(3)

\[
\|z_n - p\|^2 = \|(1 - \gamma_n)x_n + \gamma_n z_n - p\|^2 \\
\leq (1 - \gamma_n)\|x_n - p\|^2 + \gamma_n \|z_n - p\|^2 - \gamma_n(1 - \gamma_n)\phi(\|x_n - z_n\|)
\]

\[
\leq (1 - \gamma_n)\|x_n - p\|^2 + \gamma_n H^2(Tx_n, Tp) - \gamma_n(1 - \gamma_n)\phi(\|x_n - z_n\|)
\]

\[
\leq (1 - \gamma_n)\|x_n - p\|^2 - \gamma_n(1 - \gamma_n)\phi(\|x_n - z_n\|)
\]

From (1), (2) and (3), we get
(4) 
\[ \|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \alpha_n \beta_n \gamma_n (1 - \gamma_n) \varphi(\|x_n - z_n\|) \]

Therefore
\[ \alpha_n \beta_n \gamma_n (1 - \gamma_n) \varphi(\|x_n - z_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \]

This implies
\[ \sum_{n=1}^{\infty} \alpha_n \beta_n \gamma_n (1 - \gamma_n) \varphi(\|x_n - z_n\|) \leq \|x_i - p\|^2 < \infty. \]

Hence by Lemma 2.3, there exists a subsequence \( \{x_{n_k} - z_{n_k}\} \) of \( \{x_n - z_n\} \) such that 
\( \varphi(\|x_{n_k} - z_{n_k}\|) \to 0 \) as \( k \to \infty \) and hence \( \|x_{n_k} - z_{n_k}\| \to 0 \), by the continuity and strictly increasing nature of \( \varphi \). By the compactness of \( K \), we may assume that \( x_{n_k} \to q \) for some \( q \in K \). Thus
\[ \text{dist}(q,Tq) \leq \|q - x_{n_k}\| + \text{dist}(x_{n_k},Tx_{n_k}) + H(Tx_{n_k},Tq) \]
\[ \leq \|q - x_{n_k}\| + \|x_{n_k} - z_{n_k}\| + \|x_{n_k} - q\| \to 0 \quad \text{as} \quad k \to \infty. \]

Hence, \( q \) is a fixed point of \( T \). Now on taking \( q \) in place of \( p \), we get that \( \|x_n - q\| \) is a decreasing sequence by (4). Since \( \|x_n - q\| \to 0 \) as \( k \to \infty \), it follows that \( \|x_n - q\| \) decreases to 0.

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References


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