Common Fixed Point Theorems for Single and Set-valued Horizontally Weakly Compatible Pairs of Mappings in Dislocated Fuzzy Metric Spaces

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Abstract

Common fixed point theorems for two pairs of mappings are proved in dislocated fuzzy metric space using horizontally weak compatibility condition. Our results extends and generalises many existing fixed point theorems.

Keywords: Dislocated fuzzy metric space, coincidence and common fixed points, weakly compatible maps, horizontally weakly compatible maps, common (EA) Property
Since the introduction of Fuzzy Sets by Zadeh[27] in 1965, many fixed point theorems for contractions in fuzzy metric spaces and Quasi fuzzy metric spaces appeared (see[2],[4-19],[11],[14],[15],[16],[17],[20],[21],[23,25-26]). Hitzler and Seda [10] introduced the concept of dislocated metric space and studied the dislocated topologies associated with it, which is a generalisation of the conventional topologies and can be thought of as underlying the notion of dislocated metrics. They also proved a generalized version of Banach contraction mapping theorem which was applied to obtain fixed point semantics for logic programs. Later Reny George and M.S Khan [17] introduced the concept of dislocated fuzzy metric spaces and studied the associated topologies. Recently, Cho et al[2], Abbas et al[1] and Gopal et al[3] proved fixed point theorems for mappings satisfying some generalized contractive condition in Fuzzy Metric Space. The aim of our work is to prove common fixed point theorems for four mappings in a dislocated fuzzy metric space which extends and generalizes the results of Abbas et al[1], Cho et al [2], Gopal et al [3] and George and Khan [17].

2. Preliminaries

Definition 2.1 ([22]) A binary operation $\star : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $([0, 1], \star)$ is an abelian monoid with unit one such that, for all $a, b, c, d \in [0, 1]$, $a \star b \geq c \star d$ whenever $a \geq c$ and $b \geq d$.  

Definition 2.2 Let $X$ be any non empty set, $\star$ be a continuous t-norm and $M : X^2 \times [0, \infty) \to [0, 1]$ be a fuzzy set. For all $x, y, z \in X$ and $t, s \in [0, \infty)$, consider the following conditions:

$FM1 : \ M(x, y, 0) = 0$
$FM2 : \ M(x, x, t) = 1$
$FM3 : \ M(x, y, t) = 1$ and $M(y, x, t) = 1 \Rightarrow x = y$
$FM4 : \ M(x, y, t) = M(y, x, t)$
$FM5 : \ M(x, y, t + s) \geq M(x, z, t) \star M(z, y, s)$
$FM6 : \ M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous
$FM7 : \ M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous

If $M$ satisfies conditions $FM1$ to $FM6$ then $(X, M, \star)$ is called a Fuzzy Metric Space[13]. If $M$ satisfies conditions $FM1$ and $FM3$ to $FM6$ then we say that $(X, M, \star)$ is a Dislocated Fuzzy Metric Space in the sense of Kramosil and Michalek (in short $D_{KM}FMSpace$ [18]). If $M : X^2 \times (0, \infty) \to [0, 1]$ satisfies conditions $FM1$ and $FM3$ to $FM5$ and $FM7$ then we say that $(X, M, \star)$
is a Dislocated Fuzzy Metric Space in the sense of George and Veeramani (in short \(D_{GV}FMSpace\) [18]). By a Dislocated Fuzzy Metric Space (in short \(DFM – Space\) ) we mean a \(D_{KM}FMSpace\) or \(D_{GV}FMSpace\).

Let \((X, d)\) be a metric space and \(A, B, S, T : X \to X\) be mappings. Let \(C(A, S)\) denote the set of all coincidence points of the mappings \(A\) and \(S\) i.e. \(C(A, S) = \{u \in X : Su = Au\}\).

**Definition 2.3** [12] The mappings \(A\) and \(S\) are said to be weakly compatible if and only if they commute at their coincidence points.

**Definition 2.4** [19] The pairs \((A, S)\) and \((B, T)\) are said to be horizontally weakly compatible pairs iff \(u \in C(A, S), v \in C(B, T)\) and \(Au = Su = Tv = Bv \Rightarrow ASu = SAu\) and \(BTv = TBv\)

Clearly, if two pairs of mappings \((A, S)\) and \((B, T)\) are weakly compatible, then they are horizontally weakly compatible, but the converse is not necessarily true.

**Example 2.5** Let \(X = \mathbb{R}\). Define \(a \ast b = ab\) and \(M(x, y, t) = \left[ \exp \frac{|x-y| + |x| + |y|}{t} \right]^{-1}\) for all \((x, y) \in X \times X, t \in (0, \infty)\). Then \((X, M, \ast)\) is a \(D_{GV}FM – Space\).

**Definition 2.6** A sequence \(\{x_n\}\) in a \(DFM – Space\ (X, M, \ast)\) converges to \(x\) if and only if for each \(\epsilon > 0\), there exist \(n_0 \in \mathbb{N}\) such that \(M(x, x, t) > 1 - \epsilon\) for all \(n \geq n_0\) and \(t > 0\).

**Proposition 2.7** Let \((X, M, \ast)\) be a \(DFM – Space\) and \(x_n\) be a sequence in \(X\). If sequence \(x_n\) converges to \(x \in X\) then \(M(x, x, t) = 1\) for all \(t > 0\).

**Proof** We have \(M(x, x, t) \geq M(x, x_{n}, t/2) \ast M(x_{n}, x, t/2)\) for all \(n\). Taking the limit as \(n \to \infty\) we have \(M(x, x, t) \geq 1 \ast 1 = 1\).

**Lemma 2.8** In a \(DFM – Space\ (X, M, \ast)\), if \(M(x, y, qt) \geq M(x, y, t)\) for all \(x, y \in X, t \geq 0\) and \(q \in (0, 1)\) then \(x = y\).

Let \(A, B, S\) and \(T\) be self mappings of \(DFM – Space\ (X, M, \ast)\). An element \(z \in X\) is said to be a coincidence point of \(A\) and \(S\) if and only if \(Az = Sz\). \(z\) is said to be a common fixed point of \(A\) and \(S\) iff \(Az = Sz = z\).

**Definition 2.9** The pair \((A, S)\) is weakly compatible if and only if \(M(ASz, SAz, t) = 1\) for all \(z \in C(A, S)\) and \(t \in [0, \infty)\).
Definition 2.10 The pair \((A, S)\) satisfy \((EA)\) Property if and only if there exist a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ax_n = x\) for some \(x \in X\).

Definition 2.11 The pairs of mappings \((A, S)\) and \((B, T)\) are said to be horizontally weakly compatible pairs if and only if \(M(ASu, SAu, t) = 1\) and \(M(BTv, TBv, t) = 1\) for all \(t \in [0, \infty)\) whenever \(u \in C(A, S)\), \(v \in C(B, T)\) and \(Au = Su = Bv = Tv\).

Definition 2.12 The pairs of mappings \((A, S)\) and \((B, T)\) in a DFM − Space are said to satisfy common \((EA)\) property, if there exists sequence \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that \(\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = x\) for some \(x \in X\).

3. Main Results:

Consider the functions \(\phi_1 : [0, 1]^2 \to [0, 1]\) and \(\phi_2 : [0, 1]^5 \to [0, 1]\) such that for \(i = 1, 2\)
(a) \(\phi_i\) is a non-decreasing function, for all variables
(b) \(\phi_1(t, t) \geq t\), and \(\phi_2(t, t, t, t, t) \geq t\) for all \(t \in [0, \infty)\)
(c) \(\phi_i\) is continuous in all variables.

Let \(r : [0, \infty) \to [0, \infty]\) be a non-decreasing function such that \(r(\theta) < \theta\) for all \(\theta > 0\) and let

\[
U(x, y, t) = \phi_1\{M(Ax, Sx, t), M(Ax, Ty, t)\}
\]

\[
V(x, y, t) = \phi_2\{(M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, \alpha t), M(By, Sx, (2 - \alpha)t)\}.
\]

Now we present our main results as follows :

Theorem 3.1 Let \((X, M, \ast)\) be a DFM − Space. Let \(A, B, S\) and \(T\) be mappings from \(X\) into itself such that

\[
1 \leq \frac{1}{M(Ax, By, kt)} - 1 \leq r\left(\frac{1}{aU(x, y, t) + (1 - a)V(x, y, t)} - 1\right),
\]

for all \(x, y \in X\), \(a \in [0, 1]\) and \(t \in [0, \infty)\). If the pairs \(\{A, S\}\) and \(\{B, T\}\) satisfy common \((EA)\) property, \(S(X)\) and \(T(X)\) are closed subsets of \(X\), then \(C(A, S) \neq \phi\) and \(C(B, T) \neq \phi\). Further if \(\{A, S\}\) and \(\{B, T\}\) are horizontally weakly compatible pairs then \(A, B, S\) and \(T\) have a unique common fixed point.
Proof. Since the pairs \( \{A, S\} \) and \( \{B, T\} \) satisfies common \((EA)\) property, there exist sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that \( \lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} B y_n = \lim_{n \to \infty} T y_n = z \). Since, \( S(X) \) is a closed subset of \( X \), \( \lim_{n \to \infty} S x_n = z \in S(X) \). Hence, there exists \( u \in X \) such that \( Su = z \). We claim \( Au = Su \).

If not, from (3.1) we have

\[
\lim_{n \to \infty} M(Au, By_n, kt) - 1 \leq r(\lim_{n \to \infty} V(u, y_n, t) + (1-a)\lim_{n \to \infty} U(u, y_n, t) - 1) 
\]

Now we have,

\[
U(u, y_n, t) = \phi_1 \{M(Au, Su, t), M(Au, Ty_n, t)\}. \text{ Taking limit } n \to \infty, \text{ we get }
\]

\[
\lim_{n \to \infty} U(u, y_n, t) = \phi_1 \{M(Au, z, t), M(Au, z, t)\} \geq M(Au, z, t).
\]

Similarly, for \( \alpha = 1 \),

\[
V(u, y_n, t) = \phi_2 \{M(Su, Ty_n, t), M(Au, Su, t), M(By_n, Ty_n, t), M(Au, Ty_n, t), M(By_n, Su, t)\}.
\]

Taking limit \( n \to \infty \), we get

\[
\lim_{n \to \infty} V(u, y_n, t) = \phi_2 \{M(z, z, t), M(Au, z, t), M(z, z, t), M(Au, z, t)\} 
\]

\[
= \phi_2 \{1, M(Au, z, t), 1, M(Au, z, t)\} \geq M(Au, z, t). \text{ Therefore from (A), }
\]

\[
\frac{1}{M(Au, z, kt)} - 1 \leq r(\frac{1}{M(Au, z, t)} - 1) < \frac{1}{M(Au, z, t)} - 1 
\]

\[
\Rightarrow M(Au, z, kt) \geq M(Au, z, t)
\]

Hence, we have \( Au = Su = z \), i.e. \( C(A, S) \neq \phi \).

As \( T(X) \) is a closed subset of \( X \), we have,

\[
\lim_{n \to \infty} Ty_n = z \in T(X).
\]

Hence, there exist, \( w \in X \) such that \( Tw = z \).

Now we claim \( Bw = Tw \). From (4.1), we have,

\[
\frac{1}{M(Ax_n, Bw, kt)} - 1 \leq r(\frac{1}{M(Ax_n, w, t)} + (1-a)\frac{1}{M(Ax_n, w, t)} - 1)
\]

Taking limit \( n \to \infty \), and proceeding as before, we get,

\[
\frac{1}{M(z, Bw, kt)} - 1 \leq r(\frac{1}{M(z, Bw, t)} - 1)
\]
hence $Bw = Tw = z$, i.e. $C(B, T) \neq \phi$.

Thus we have $Au = Su = Bw = Tw = z$. Since the pairs $(A, S)$ and $(B, T)$ are horizontally weakly compatible pairs, we have, $M(ASu, SAu, t) = 1$ and $M(BTw, TBw, t) = 1$, i.e $M(Az, Sz, t) = 1$ and $M(Bz, Tz, t) = 1$. Hence $Az = Sz$ and $Bz = Tz$. Now we claim that $Az = z$. From (3.1) we have

$$\frac{1}{M(Az, Bw, kt)} - 1 \leq r\left(\frac{1}{aU(z, w, t) + (1-a)V(z, w, t)} - 1\right) \quad \text{(B)}$$

$$U(z, w, t) = \phi_1\{M(Az, Sz, t), M(Az, Tw, t)\}$$

$$= \phi_1\{1, M(Az, z, t)\} \geq M(Az, z, t)$$

Similarly, for $\alpha = 1$,

$$V(z, w, t) = \phi_2\{M(Sz, Tw, t), M(Az, Sz, t), M(Bw, Tw, t), M(Az, Tw, t), M(Bw, Sz, t)\}$$

$$= \phi_2\{1, M(Az, z, t), M(Az, z, t), M(z, Az, t)\} \geq M(Az, z, t).$$

Hence from (B) we get $\frac{1}{M(Az, Bw, kt)} - 1 \leq r\left(\frac{1}{M(Az, z, t)} - 1\right)$ which implies $Az = z$.

Similarly it can be shown that $z = Bz = Tz$. Hence, $z$ is a common fixed point of $A, B, S$ and $T$. Now we prove that $z$ is the unique common fixed point of $A, B, S$ and $T$. If not, let there exist another common fixed point $p$ of $A, B, S$ and $T$ such that $p \neq z$. Since the pairs $(A, S)$ and $(B, T)$ are horizontally weakly compatible, we get $M(ASp, SAp, t) = 1$ and $M(BTp, TBp, t) = 1$, i.e. $M(Ap, Sp, t) = 1$ and $M(Bp, Tp, t) = 1$. From (3.1) we have,

$$\frac{1}{M(p, z, kt)} - 1 \leq r\left(\frac{1}{aU(p, z, t) + (1-a)V(p, z, t)} - 1\right) \quad \text{(C)}$$

$$U(p, z, t) = \phi_1\{M(Ap, Sp, t), M(Ap, Tz, t)\} = \phi_1\{1, M(p, z, t)\} \geq M(p, z, t)$$

Similarly, for $\alpha = 1$,

$$V(p, z, t) = \phi_2\{M(Sp, Tz, t), M(Ap, Sp, t), M(Bz, Tz, t), M(Ap, Tz, t), M(Bz, Sp, t)\}$$

$$= \phi_2\{1, M(p, z, t), M(p, z, t), M(z, p, t)\} \geq M(p, z, t).$$

Hence from (C) we get $\frac{1}{M(p, z, kt)} - 1 \leq r\left(\frac{1}{M(p, z, t)} - 1\right)$ which implies $p = z$. Thus $z$ is the unique common fixed point of $A, B, S$ and $T$. 

$\Box$
Since every fuzzy metric space is a $DFM - Space$, as a direct consequence of Theorem 3.1 we have the following:

**Theorem 3.2** Let $(X, M, \ast)$ be a fuzzy metric space, $A$, $B$, $S$ and $T$ be mappings from $X$ into itself satisfying (3.1). If the pairs \{A, S\} and \{B, T\} satisfy common(EA) property, $S(X)$ and $T(X)$ are closed subsets of $X$, then $C(A, S) \neq \phi$ and $C(B, T) \neq \phi$. Further if \{A, S\} and \{B, T\} are horizontally weakly compatible pairs then $A$, $B$, $S$ and $T$ have a unique common fixed point.

**Remark 3.3** Theorem 3.1 and 3.2 are applicable to a larger class of mappings than those given in [1], [2] and [3] as is evident from the following examples.

**Example 3.4** Let $X = [0, 100]$ and $a \ast b = ab$. Let $M$ be the Fuzzy Metric induced by $d$, where $d(x, y) = |x - y| + \frac{|x| + |y|}{2}$, for all $x, y \in X$. Then $(X, M, \ast)$ is a Dislocated Fuzzy Metric space but not a fuzzy metric space. Define self maps $A, B, S, T$ on $X$ as follows:

$$Ax = \begin{cases} 0, & x = 0 \\ 4, & x > 0 \end{cases}$$

$$Bx = \begin{cases} 0, & x = 0 \\ 10, & x > 0 \end{cases}$$

$$Sx = \begin{cases} 0, & x = 0 \\ 24, & x > 0 \end{cases}$$

$$Tx = \begin{cases} 0, & x = 0 \\ 24, & 0 < x \leq 50 \\ 10, & x > 50 \end{cases}$$

Let $\phi_2(x_1, x_2, x_3, x_4, x_5) = \min(x_1, x_2, x_3, x_4, x_5)$. Then we see that $A, B, S$ and $T$ satisfy (4.1) with $k = \frac{1}{2}, a = 0, \alpha = \frac{1}{2}$ and $r(\theta) = p\theta$, where $p \in [0, 1)$. We have, $C(A, S) = \{0\}$ and $C(B, T) = \{0\} \cup (50, 100]$ and $Au = Su = Bu = Tv$ only for $u = v = 0$. Clearly $M(A\emptyset 0, S\emptyset 0, t) = 1$, $M(BT\emptyset 0, TB\emptyset 0, t) = 1$, and so the pairs $(A, S)$ and $(B, T)$ are horizontally weakly compatible. We see that mappings $A, B, S$ and $T$ satisfies the hypothesis of Theorem 4.5 and 0 is the unique common fixed point of $A, B, S$ and $T$.

**Example 3.5** In Example 3.4 if we take $d(x, y) = |x - y|$, then $(X, M, \ast)$ is a fuzzy metric space. We see that the mappings $A, B, S$ and $T$ satisfies the
hypothesis of Theorem 3.2 and 0 is the unique common fixed point.

4. IMPLICIT FUNCTIONS AND COMMON FIXED POINT

Let $\psi$ denote the family of all continuous functions $F : [0, 1]^6 \to R$, satisfying the following conditions:

$F(u(kt), 1, u(t), 1, 1, u(t)) \geq 0$ or $F(u(kt), 1, 1, u(t), u(t), 1) \geq 0$ or

$F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 0 \Rightarrow u(kt) > u(t)$

where $u : [0, \infty] \to [0, 1]$ and $t > 0$.

Some Examples of functions of class $\psi$ are as follows:

Example: 4.1 Define $F : [0, 1]^6 \to R$ as $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi\{\min(t_2, t_3, t_4, t_5, t_6)\}$ where $\phi : [0, 1] \to [0, 1]$ is a continuous function such that $\phi(t) > t$ for $0 < t < 1$

Example: 4.2 Define $F : [0, 1]^6 \to R$ as $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k\{\min(t_2, t_3, t_4, t_5, t_6)\}$ where $k > 1$

Example: 4.3 Define $F : [0, 1]^6 \to R$ as $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - kt_2 - \min(t_3, t_4, t_5, t_6)$ where $k > 0$

Example: 4.4 Define $F : [0, 1]^6 \to R$ as $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^{1/3} - kt_2 t_3 t_4 t_5 t_6$ where $k > 1$

Example: 4.5 Define $F : [0, 1]^6 \to R$ as $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 t_5 - bt_3 - ct_4 t_6$ where $(a + c) > 1$ or $(b + c) > 1$

Example: 4.6 Define $F : [0, 1]^6 \to R$ as $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi(t_2, t_3, t_4, t_5, t_6)$ where $\phi : [0, 1]^5 \to [0, 1]$ is a non-decreasing function such that $\phi(t, t, t, t, t) > t$

Theorem 4.7 Let $(X, M, *)$ be a DFM- Space. $A, B, S$ and $T$ be mappings from $X$ into itself such that

\begin{equation}
(4.1)
F(M(Ax, By, kt), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t), M(Sx, By, t), M(Ty, Ax, t), ) \geq 0
\end{equation}

for all $x, y \in X,, t > 0, 0 < k < 1$ and $F \in \psi$. Suppose the pairs $\{A, S\}$ and $\{B, T\}$ satisfy common (EA) property, $T(X)$ and $S(X)$ are closed subsets of $X$ and the pairs $\{A, S\}$ and $\{B, T\}$ are horizontally weakly compatible. Then $A, B, S,$ and $T$ have a unique common fixed point in $X$.

Proof. Since the pairs $\{A, S\}$ and $\{B, T\}$ satisfies common (EA) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n =$
\[\lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z\] for some \(z \in X\). Since \(S(X)\) is a closed subset of \(X\), there exist \(u \in X\) such that \(Su = z\). From (4.1) we have,

\[F(M(Au, By_n, kt), M(Su, Ty_n, t), M(Su, Au, t)M(Ty_n, By_n, t), M(Su, By_n, t), M(Ty_n, Au, t), ) \geq 0\]

As \(n \to \infty\), we get

\[F(M(Au, z, kt), M(z, z, t), M(z, Au, t), M(z, z, t), M(z, Au, t)) \geq 0\]

i.e. \(F(M(Au, z, kt), 1, M(Au, z, t), 1, M(Au, z, t)) \geq 0\), which implies

\(M(Au, z, kt) \geq M(Au, z, t)\). Hence \(Au = z = Su\), i.e. \(C(A, S) \neq \phi\).

Again since \(T(X)\) is a closed subset of \(X\) there exist \(w \in X\) such that \(Tw = z\). Now we claim \(Bw = Tw\). From (4.1), we have,

\[F(M(Ax_n, Bw, kt), M(Sx_n, Tw, t), M(Sx_n, Ax_n, t)M(Tw, Bw, t), M(Sx_n, Bw, t), M(Tw, Ax_n, t), ) \geq 0\]

As \(n \to \infty\), we get

\[F(M(z, Bw, kt), M(z, z, t), M(z, Bw, t), M(z, Bw, t), M(z, z, t) \geq 0\]

i.e. \(F(M(z, Bw, kt), 1, 1, M(z, Bw, t), M(z, Bw, t), 1) \geq 0\)

i.e., \(M(z, Bw, kt) \geq M(z, Bw, t)\) for all \(t > 0\). Hence \(Bw = z = Tw\), i.e., \(C(B, T) \neq \phi\). Thus we have \(Au = Su = Bw = Tw = z\). Since the pairs \((A, S)\) and \((B, T)\) are horizontally weakly compatible, we have, \(M(ASu, SAu, t) = 1\) and \(M(BTw, Tbw, t) = 1\), i.e \(M(Az, Sz, t) = 1\) and \(M(Bz, Tz, t) = 1\). Hence \(Az = Sz\) and \(Bz = Tz\). Now we claim \(Az = z\) From (4.1) we have,

\[F(M(Az, Bw, kt), M(Sz, Tw, t), M(Sz, Az, t), M(Tw, Bw, t), M(Sz, Bw, t), M(Tw, Az, t), ) \geq 0\]

i.e.\(F(M(Az, z, kt), M(z, z, t), M(z, Az, t), M(z, z, t), M(z, z, t), M(z, Az, t)) \geq 0\)

i.e.\(F(M(Az, z, kt), 1, M(Az, z, t), 1, 1, M(Az, z, t)) \geq 0\) which implies \(M(Az, z, kt) \geq M(Az, z, t)\) and so \(Az = z\). Thus we have \(Az = Sz = z\). Similarly \(Bz = Tz = z\) and so \(z\) is the common fixed point of \(A, B, S\) and \(T\). Now we prove that \(z\) is unique common fixed point. If not, let there exist another common fixed point \(p\) of \(A, B, S\) and \(T\) such that \(p \neq z\). From (4.1), we have
\[ F(M(Ap, Bz, kt), M(Sp, Tz, t), M(Sp, Ap, t), M(Tz, Bz, t), M(Sp, Bz, t), M(Tz, Ap, t)) \geq 0 \]

i.e. \[ F(M(Ap, z, kt), M(Ap, z, t), 1, 1, M(Ap, z, t), M(Ap, z, t)) \geq 0 \]

i.e. \( M(Ap, z, kt) \geq M(Ap, z, t) \) which implies \( M(p, z, kt) \geq M(p, z, t) \) and so \( p = z \). Thus \( z \) is the unique common fixed point of \( A, B, S \) and \( T \).

\[ \square \]

**Corollary 4.8** Let \((X, M, *)\) be a DFM – Space. \( A, B, S \) and \( T \) be mappings from \( X \) into itself satisfying any of the following conditions:

1. \( M(Ax, By, kt) \geq \phi(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t), M((Sx, By, t), M(Ty, Ax, t))\}) \) where \( \phi : [0, 1] \to [0, 1] \) is a continuous function such that \( \phi(t) > t \) for \( 0 < t < 1 \)

2. \( M(Ax, By, kt) \geq k(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t), M((Sx, By, t), M(Ty, Ax, t))\}) \) where \( k > 1 \)

3. \( M(Ax, By, kt) \geq k(M(Sx, Ty, t)) + (\min\{M(Sx, Ax, t), M(Ty, By, t), M((Sx, By, t), M(Ty, Ax, t))\}) \) where \( k > 1 \)

4. \( M(Ax, By, kt) \geq kM(Sx, Ty, t)M(Sx, Ax, t)M(Ty, By, t)M((Sx, By, t)M(Ty, Ax, t) \)

where \( k > 1 \)

5. \( M(Ax, By, kt) \geq a(M(Sx, Ty, t)M((Sx, By, t)) + bM(Sx, Ax, t) + c(M(Ty, By, t)M(Ty, Ax, t)) \) where \( (a + c) > 1 \) or \( (b + c) > 1 \)

6. \( M(Ax, By, kt) \geq \phi(M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t), M((Sx, By, t), M(Ty, Ax, t)) \) where \( \phi : [0, 1]^5 \to [0, 1] \) is a non-decreasing function such that \( \phi(t, t, t, t, t) > t \)

If the pairs pairs \((A, S)\) and \((B, T)\) satisfy common \((EA)\) property and are horizontally weakly compatible, then \( A, B, S, \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** The Proof of corollaries corresponding to the above contractive conditions follows from Theorem (4.1) above and the Examples (4.1) to (4.6).

\[ \square \]

**Remark 4.9** Corollary 4.8 corresponding to condition (i) is a generalised version of Theorem 2.1 of M Abbas et al [1] whereas conditions (ii) to (vi) corresponding to various conditions presents a sharpened form of Corollary
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5. COMMON FIXED POINT THEOREMS FOR SET VALUED MAPPINGS IN A DFM SPACE

In this section we establish fixed point results for Set valued mappings in a DFM Space. In the first place we consider the following definitions:

Let \((X, M, t)\) be the dislocated metric space and \(CB(X)\) denote the set of all non-empty closed and bounded subsets of \(X\). For \(A, B \in CB(X)\) and for every \(t > 0\), denote \(H(A, B, t) = \sup\{M(a, b, t); a \in A, b \in B\}\) and \(\delta_M(A, B, t) = \inf\{M(a, b, t); a \in A, b \in B\}\).

**Remark 5.1.** If \(A\) consists of a single point \(a\), we write \(\delta_M(A, B, t) = M(a, B, t)\).

If \(B\) also consists of a single point \(b\), then \(\delta_M(A, B, t) = M(a, b, t)\).

Clearly we have \(\delta_M(A, B, t) = \delta_M(B, A, t) \geq 0\) and \(\delta_M(A, B, t) = 1 \iff A = B = \{a\}\) for all \(A, B \in CB(X)\).

**Definition 5.2.** A point \(x \in X\) is called a coincidence point of \(A : X \rightarrow X\), \(B : X \rightarrow CB(X)\) if \(Ax \in Bx\).

**Definition 5.3.** Maps \(A : X \rightarrow X\) and \(B : X \rightarrow CB(X)\) are said to be compatible if \(ABx \in CB(X)\) for all \(x \in X\) and \(\lim_{n \rightarrow \infty} H(ABx_n, BAx_n, t) = 1\) whenever \(\{x_n\}\) is a sequence in \(X\) such that \(Bx_n \rightarrow M \in CB(x)\) and \(Ax_n \rightarrow x \in \mathbb{X}\).

**Definition 5.4.** Maps \(A : X \rightarrow X\) and \(B : X \rightarrow CB(X)\) are said to be weakly compatible if they commute at coincidence points, i.e., if \(ABx = BAx\) whenever \(Ax \in Bx\).

**Definition 5.5.** Let \(A, B : X \rightarrow X\) and \(S, T : X \rightarrow CB(X)\) be single and set valued mappings respectively. The pairs of mappings \((A, S)\) and \((B, T)\) in a DFM – Space are said to be Horizontally weakly compatible pairs if and only if \(u \in C(A, S), v \in C(B, T)\) and \(Au \in Su\) and \(Bv \in Tv \Rightarrow M(ASu, SAu, t) = 1\) and \(M(BTv, TBv, t) = 1\) for all \(t \in [0, \infty)\).

**Definition 5.6.** Mappings \(A, B, S\) and \(T\) in a DFM – Space are said to satisfy common (EA) Property, if there exists sequence \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that \(\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = x\) for some \(x \in X\).

Now we present our results as under: Consider the functions \(\phi_1 : [0, 1]^2 \rightarrow [0, 1]\) and \(\phi_2 : [0, 1]^5 \rightarrow [0, 1]\) such that for \(i = 1, 2\) (a) \(\phi_i\) is a non-decreasing function, for all variables.
(b) $\phi_1(t, t) \geq t$, and $\phi_2(t, t, t, t, t) \geq t$ for all $t \in [0, \infty)$
(c) $\phi_i$ is continuous in all variables.

**Theorem 5.7.** Let $(X, M, t)$ be a Dislocated Fuzzy Metric Space with $A, B : X \to X, S, T : X \to CB(X)$ be single and set valued mappings respectively such that the maps $(A, S)$ and $(B, T)$ are horizontally weakly compatible and satisfy the inequality

$$
\int_0^{\delta_M(Sx, Ty, kt)} \phi(t)dt > \int_0^{m(x, y, t)} \phi(t)dt
$$

where $k \in (0, 1)$, $\phi$ is a function which is sum able, Lebseque integrable, non-negative and such that $\int_0^t \phi(t)dt > 0$ for each $\epsilon > 0$ where

$$
m(x, y, t) = a\phi_1\{M(Ax, Sx, t), M(Ax, Ty, t)\} + (1 - a)\phi_2\{(M(Sx, Ty, t), H(Ax, Sx, t), H(By, Ty, t), H(Ax, Ty, \alpha t), H(By, Sx, (2 - \alpha t))\}
$$

for every $x, y \in X, t \neq 0, \alpha \in (0, 2)$. Then $A, B, S, T$ have a unique common fixed point in $X$.

**Proof:** Since the pair $\{A, S\}$ and $\{B, T\}$ satisfies the common (EA) property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z \in M
$$

where $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ty_n = M \in CB(X)$. Since $S(X)$ is a closed bounded subset of $X$, $\lim_{n \to \infty} Sx_n = z \in S(X)$. Hence, there exists $u \in X$ such that $Su = z$.

Now we claim that $Au \in Su$. Let us assume otherwise. Then, substituting $x = Au$ and $y = By_n$ in (A), we get,

$$
lim_{n \to \infty} m(Au, By_n, kt) = a\phi_1\lim_{n \to \infty}\{M(AAu, SAu, t), M(AAu, TBBy_n, t)\} + \phi_2\lim_{n \to \infty}\{(M(SAu, TBBy_n, t), H(AAu, SAu, t), H(BBy_n, TBBy_n, t), H(AAu, TBBy_n, t), H(BBy_n, SAu, t)\}
$$

$$
= a\phi_1\{M(AAu, SAu, t), M(AAu, TBv, t)\} + (1 - a)\phi_2\{M(SAu, TBv, t), H(AAu, SAu, t), H(BTv, TBv, t), H(AAu, TBv, t), H(BTv, SAu, t)\}
$$

> $a\phi_1\{\delta_M(SAu, SAu, t), \delta_M(SAu, TBv, t)\}

$$
= a\phi_1\{1, \delta_M(SAu, TBv, t)\} + (1 - a)\phi_2\{\delta_M(SAu, TBv, t), 1, 1, \delta_M(SAu, TBv, t)\}
$$

> $\delta_M(SAu, TBv, t)

$$
\int_0^{\delta_M(SAu, TBv, t)} \phi(t)dt > \int_0^{m(Au, By_n, t)} \phi(t)dt > \int_0^{\delta_M(SAu, TBv, t)} \phi(t)dt
$$

which is a contradiction. So we have $Au \in Su \Rightarrow C(A, S) \neq \phi$ similarly we can show that $Bu \in Tv \Rightarrow C(B, T) \neq \phi$ Since the pairs $(A, S)$ and $(B, T)$ are horizontally weakly compatible pairs, we have,

$M(ASu, Szu, t) = 1$ and $M(BTv, TBw, t) = 1$, i.e $M(Az, Sz, t) = 1$ and $M(Bz, Tz, t) = 1$.

i.e. $Az = Sz$ and $Bz = Tz$

Now we claim $Az = z$. If not, we have $\delta_M(SAz, Tz, t) < 1$
Taking $x = Az$ and $y = z$ and $\alpha = 1$ in (A) we have,

$$m(Az, z, t) = a \phi_1 \{M(AAz, SAz, t), M(AAz, Tz, t)\} + (1 - a) \phi_2 \{M(ASz, Tz, t), H(AAz, SAz, t), H(Bz, Tz, t), H(AAz, Tz, t), H(Bz, SAz, t)\}$$

$$\geq a \phi_1 \{M(AAz, SAz, t), M(AAz, Tz, t)\} + (1 - a) \phi_2 \{M(ASz, Tz, t), M(AAz, SAz, t), M(AAz, Tz, t), M(Bz, SAz, t)\}$$

$$\geq a \phi_1 \{1, \delta_M(SAz, Tz, t)\} + (1 - a) \phi_2 \{\delta_M(SAz, Tz, t), 1, 1, \delta_M(SAz, Tz, t), M(SAz, Tz, t)\}$$

$$m(Az, z, t) \geq \delta_M(SAz, Tz, t)$$

Thus we have,

$$\int_0^{\delta_M(SAz, Tz, t)} \phi(t) dt > \int_0^{m(Az, z, t)} \phi(t) dt > \int_0^{\delta_M(SAz, Tz, t)} \phi(t) dt$$

which is a contradiction. So $Az = z$.

Similarly we can show $Bz = Tz = z$.

Thus $z$ is the common fixed point of $A$, $B$, $S$ and $T$.

For uniqueness, suppose $u \neq z$ be another fixed point of $A$, $B$, $S$ and $T$.

Then,

$$m(z, u, t) = a \phi_1 \{M(Az, Sz, t), M(Az, Tu, t)\} + (1 - a) \phi_2 \{(M(Sz, Tu, t), H(Az, Sz, t), H(Bu, Tu, t), H(Az, Tu, \alpha t), H(Bu, Sz, (2 - \alpha) t))\}$$

Letting $\alpha = 1$ we have,

$$m(z, u, t) = a \phi_1 \{M(Az, Sz, t), M(Az, Tu, t)\} + (1 - a) \phi_2 \{(M(Sz, Tu, t), M(Az, Sz, t), M(Bu, Tu, t), M(Az, Tu, t), M(Bu, Sz, t))\}$$

$$\geq a \phi_1 \{1, \delta_M(Sz, Tu, t)\} + (1 - a) \phi_2 \{\delta_M(Sz, Tu, t), 1, 1, \delta_M(Sz, Tu, t)\}$$

Thus we have,

$$\int_0^{\delta_M(Sz, Tu, t)} \phi(t) dt > \int_0^{m(z, u, t)} \phi(t) dt > \int_0^{\delta_M(Sz, Tu, t)} \phi(t) dt$$

which implies, $Sz = Tu \Rightarrow z = u$

This completes the proof.

**Corollary 5.8.** Let $(X, M, t)$ be a Dislocated Fuzzy Metric Space with $A, B : X \rightarrow X, S, T : CB(X)$ be single and set valued mappings respectively such that the maps $(A, S)$ and $(B, T)$ are horizontally weakly compatible and satisfy the inequality

$$m(x, y, t) \geq \delta_M(Sx, Ty, t)$$

where $k \in (0, 1)$, $\phi$ is a function which is sum able, Lebesgue integrable, non-negative and such that $\int_0^\epsilon \phi(t) dt > 0$ for each $\epsilon > 0$. Where $m(x, y, t) = \min\{(M(Sx, Ty, t), H(Ax, Sz, t), H(By, Ty, t), H(Ax, Ty, \alpha t), H(By, Sz, (2 - \alpha) t))\}$ for every $x, y \in X$, $t \neq 0, \alpha \in (0, 2)$. Then $A$, $B$, $S$, $T$ have a unique common fixed point in $X$.

**Proof:** Substituting $a = 0$ and proceeding as before, we can show that $A$, $B$, $S$ and $T$ have unique fixed point in $X$.

**Remark 5.9.** The above corollary is the main theorem of Sumitra et al[24], is a special cases of our main result in dislocated fuzzy metric spaces.
More research is being undertaken in the area of Fuzzy Metric space, Dislocated Fuzzy Metric Space, Dislocated Quazi Fuzzy Metric Space etc., by including more contractive conditions including various types of mappings.

References


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