Some New Hermite-Hadamard-like Type Inequalities on Geometrically Convex Functions

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Abstract

In this article, by using the Hölder integral inequality, power mean integral inequality and the property of modulus, some new upper bounds for twice differentiable functions whose $q$-th powers are geometrically convex and monotonically decreasing are obtained.

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1 Introduction

The following definition for convex functions is well known in the mathematical literature:

Definition 1. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on $I$ if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$  \hspace{1cm} (1)

holds for all $x, y \in I$ and $t \in [0, 1]$, and $f$ is said to be concave on $I$ if the inequality (1) holds in reversed direction.
Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard’s inequality, due to its rich geometrical significance and applications, which is stated as follow:

\[ f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x)dx \leq \frac{f(a) + f(b)}{2}. \] (2)

The double inequalities (2) hold in reversed direction if \( f \) is concave.

The inequality of Hermite-Hadamard type has been considered the most useful inequality in mathematical analysis. Some of the classical inequalities for means can be derived from (2) for particular choices of the function \( f \). The above definition has opened up the most extended, useful and multidisciplinary domain od mathematics, namely, convex analysis.

For more results on Hermite-Hadamard-type inequality providing new proofs, noteworthy extensions, generalization and numerous applications for convex functions, see \([1, 2, 3, 7, 8, 9, 10, 11, 13]\) and references therein.

In \([13]\), Özdemir introduced the concept of geometrically convex functions as following:

**Definition 2.** A function \( f : I \subseteq R_+ \rightarrow R_+ \) is said to be geometrically convex if the following inequality

\[ f(a^t b^{1-t}) \leq [f(a)]^t [f(b)]^{1-t} \]

holds, for all \( a, b \in I \) and \( t \in [0, 1] \).

For some recent results connected with geometrically convex functions, you may see \([4, 5, 6, 11, 12, 13]\).

**Definition 3.** Let \( a, b \in R \) with \( a, b \neq 0 \) and \( |a| \neq |b| \). The logarithmic mean \( L(\cdot, \cdot) \) for real numbers was introduced as follows:

\[ L(a, b) = \frac{b - a}{\ln |b| - \ln |a|}. \]

In \([4]\), Özdemir established the following theorems for differentiable functions whose \( q \)-th powers are geometrically convex and monotonically decreasing:

**Theorem 1.1.** Let \( f : I \subseteq R_+ \rightarrow R_+ \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is geometrically convex and monotonically decreasing on \([a, b]\), then the
Theorem 1.4. If \( C \) holds, where \( \text{interior} \ 1^p > 1 \), then the following inequality holds:

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{(b-x)f(b) + (x-a)f(a)}{b-a} \right| 
\leq \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ \frac{(x-a)^2}{b-a} L^\frac{1}{2} \left( |f'(x)|^q, |f'(a)|^q \right) \right. 
\left. + \frac{(b-x)^2}{b-a} L^\frac{1}{2} \left( |f'(x)|^q, |f'(b)|^q \right) \right]
\]

holds, where \( 1 < p < \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( L(\cdot, \cdot) \) is logarithmic mean for real numbers.

Theorem 1.2. Let \( f : I \subseteq R_+ \rightarrow R_+ \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a,b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is geometrically convex and monotonically decreasing on \([a,b]\) for \( q \geq 1 \), then the following inequality holds:

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{(b-x)f(b) + (x-a)f(a)}{b-a} \right| 
\leq \left( \frac{1}{2} \right)^{\frac{1}{q}} \left[ \frac{(x-a)^2}{b-a} \left| f'(a) \right| \mu_0^k(k) + \frac{(b-x)^2}{b-a} \left| f'(b) \right| \mu_0^l(l) \right]
\]

holds, where

\[
k = \left| \frac{f'(x)}{f'(a)} \right|, \quad l = \left| \frac{f'(x)}{f'(b)} \right|,
\mu_0(\alpha) = \frac{\alpha - \ln \alpha - 1}{(\ln \alpha)^2}.
\]

We recall the well-known Hölder’s integral inequality which can be stated as follows:

Theorem 1.3. If \( f(x) \geq 0, g(x) \geq 0 \) and \( f(x) \in L^p[a,b], g(x) \in L^q[a,b] \) and \( p > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left( \int_a^b f(x)g(x)dx \right) \leq \left( \int_a^b f^p(x)dx \right)^\frac{1}{p} \left( \int_a^b g^q(x)dx \right)^\frac{1}{q} \quad (3)
\]

In [1], Changjian and Bencze proved the following theorem:

Theorem 1.4. If \( f(x) \geq 0, g(x) \geq 0 \) and \( f(x) \in L^p[a,b], g(x) \in L^q[a,b] \) and \( p > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left( \int_a^b f(x)g(x)dx \right) \leq C(p,t) \left( \int_a^b f^p(x)dx \right)^\frac{1}{p} \left( \int_a^b g^q(x)dx \right)^\frac{1}{q} \quad (4)
\]

holds, where \( C(p,t) = \frac{1}{p}t^{\frac{1}{p}-1} + (1 - \frac{1}{p})t^{\frac{1}{q}} \).
In this article, the main aim is to establish some new upper bounds for twice differentiable functions whose $q$-th powers are geometrically convex and monotonically decreasing.

2 Main results

In order to prove our main theorems, we need the following lemma:

**Lemma 1.** Let $f : I \to R$ be a twice differentiable function on the interior $I^0$ of an interval $I$ in $R$ where $a, b \in I$ with $a < b$. If $f'' \in L_1[a, b]$, then for $t \in [0, 1]$ the following identity holds:

$$I(f)(a, b) = \frac{1}{b - a} \int_a^b f(u)du - \frac{(b - x)f(b) + (x - a)f(a)}{b - a}$$

$$+ \frac{(b - x)^2 f'(b) - (x - a)f'(a)}{2(b - a)}$$

$$= \frac{(x - a)^3}{2(b - a)} \int_0^1 t^2 f''(ta + (1 - t)x)dt$$

$$+ \frac{(x - b)^3}{2(b - a)} \int_0^1 t^2 f''(tb + (1 - t)x)dt.$$

**Proof.** By integration by parts, we get the desired result.

We will use this lemma for obtaining several following results for twice differentiable functions similar to Theorem 1.1 and Theorem 1.2 from [4].

**Theorem 2.1.** Let $f : I \subset R_+ \to R_+$ be a twice differentiable function on the interior $I^0$ of an interval $I$ in $R$ and $f'' \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If $|f''|$ is geometrically convex and monotonically decreasing on $[a, b]$, then the following inequality holds,

$$|I(f)(a, b)| \leq \frac{(x - a)^3}{2(b - a)} \mu_1\left(\frac{|f''(a)|}{|f''(x)|}, k_1\right) + \frac{(b - x)^3}{2(b - a)} \mu_1\left(\frac{|f''(b)|}{|f''(x)|}, k_2\right)$$

holds, where

$$k_1 = \frac{|f''(a)|}{|f''(x)|}, \quad k_2 = \frac{|f'(b)|}{|f'(x)|},$$

$$\mu_1(\alpha, \beta, \gamma) = \frac{\alpha \{\ln \gamma (\ln \gamma - 2)\} + 2\beta (\gamma - 1)}{(\ln \gamma)^3}.$$
Proof. Using Lemma 1, we get
\[
|I(f)(a, b)| \leq \left| \frac{(x - a)^3}{2(b - a)} \int_0^1 t^2 \left| f''(ta + (1 - t)x) \right| dt \right. \\
+ \left. \frac{(x - b)^3}{2(b - a)} \int_0^1 t^2 \left| f''(tb + (1 - t)x) \right| dt. \tag{5} \right.
\]

Note that, for \( t \in [0, 1] \), we have \( a^t b^{1-t} \leq ta + (1 - t)b \).

Since \( |f''| \) is geometrically convex and monotonically decreasing on \([a, b]\), for \( t \in [0, 1] \) we have that
\[
\left| f''(ta + (1 - t)x) \right| \leq \left| f''(a^t x^{1-t}) \right| \leq \left| f''(a) \right| \left| f''(x) \right|^{1-t},
\]
\[
\left| f''(tb + (1 - t)x) \right| \leq \left| f''(b^t x^{1-t}) \right| \leq \left| f''(b) \right| \left| f''(x) \right|^{1-t}. \tag{6} \]

Therefore we obtain
\[
(a) \int_0^1 t^2 \left| f''(ta + (1 - t)x) \right| dt \\
\leq \int_0^1 t^2 \left| f''(a^t x^{1-t}) \right| dt \\
\leq \int_0^1 t^2 \left\{ \left| f''(a) \right| \left| f''(x) \right|^{1-t} \right\} dt \\
= \mu_1 \left( \left| f''(a) \right|, \left| f''(x) \right|, k_1 \right) \tag{7} \]
and
\[
(b) \int_0^1 t^2 \left| f''(tb + (1 - t)x) \right| dt \leq \mu_1 \left( \left| f''(b) \right|, \left| f''(x) \right|, k_2 \right) \tag{8}.
\]

By substituting (7)-(8) in (5), we easily get the desired result.

Corollary 2.1. If we choose \( |f''(a)| = |f''(b)| \) in Theorem 2.1, then we obtain:
\[
|I(f)(a, b)| \leq \left\{ \frac{(x - a)^3 + (b - x)^3}{2(b - a)} \right\} \mu_1 \left( \left| f''(a) \right|, \left| f''(x) \right|, k_1 \right).
\]
Theorem 2.2. Let \( f : I \subset \mathbb{R}_+ \to \mathbb{R}_+ \) be a twice differentiable function on the interior \( I^0 \) of an interval \( I \) in \( \mathbb{R} \) and \( f'' \in L[a, b] \), where \( a, b \in I \) with \( a < b \). If \( |f''|^q \) is geometrically convex and monotonically decreasing on \([a, b]\) for \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
|I(f)(a, b)| \leq \left( \frac{1}{2p+1} \right)^\frac{1}{q} \left( \frac{(x-a)^3}{2(b-a)} + \frac{(b-x)^3}{2(b-a)} \right) C(p, l) \left( \int_0^1 \left| f''(ta + (1-t)x) \right|^{q} \ dt \right)^\frac{1}{q}. 
\]

where \( C(p, l)(i = 1, 2) \) are defined as in Theorem 1.4 and \( L(\cdot, \cdot) \) is logarithmic mean for real numbers.

Proof. From Lemma 1 and using Theorem 1.4, we get

\[
|I(f)(a, b)| \leq \left( \frac{1}{2p+1} \right)^\frac{1}{q} \left( \frac{(x-a)^3}{2(b-a)} C(p, l_1) \left( \int_0^1 \left| f''(ta + (1-t)x) \right|^{q} \ dt \right)^\frac{1}{q} 
\]

\[
+ \frac{(b-x)^3}{2(b-a)} C(p, l_2) \left( \int_0^1 \left| f''(tb + (1-t)x) \right|^{q} \ dt \right)^\frac{1}{q}. 
\]

\[
= \left( \frac{1}{2p+1} \right)^\frac{1}{q} \left[ \frac{(x-a)^3}{2(b-a)} C(p, l_1) \left( \int_0^1 \left| f''(ta + (1-t)x) \right|^{q} \ dt \right)^\frac{1}{q} 
\]

\[
+ \frac{(b-x)^3}{2(b-a)} C(p, l_2) \left( \int_0^1 \left| f''(tb + (1-t)x) \right|^{q} \ dt \right)^\frac{1}{q}. \tag{9}
\]

Note that, for \( t \in [0, 1] \), we have \( a^t b^{1-t} \leq ta + (1-t)b \).

Since \( |f''| \) is geometrically convex and monotonically decreasing on \([a, b]\), for \( t \in [0, 1] \) we have that

\[
|f''(ta + (1-t)x)|^{q} \leq \left| f''(a^t x^{1-t}) \right|^{q} \leq \left( \left| \frac{f''(a)}{t} \right| \left| f''(x) \right|^{1-t} \right)^{q},
\]

\[
|f''(tb + (1-t)x)|^{q} \leq \left| f''(b x^{1-t}) \right|^{q} \leq \left( \left| \frac{f''(b)}{t} \right| \left| f''(x) \right|^{1-t} \right)^{q}. \tag{10}
\]

Therefore, by (10) we obtain

\[
(a) \int_0^1 \left| f''(ta + (1-t)x) \right|^{q} \ dt
\]
\[ \begin{align*}
&\leq \int_0^1 \left| f''(a^t x^{1-t}) \right|^q dt \\
&\leq \int_0^1 \left\{ \left| f''(a) \right|^t \left| f''(x) \right|^{1-t} \right\}^q dt \\
&= L \left( \left| f''(a) \right|^q, \left| f''(x) \right|^q \right), \\
&= \left( b \right) \int_0^1 \left| f''(tb + (1-t)x) \right|^q dt \leq L \left( \left| f''(b) \right|^q, \left| f''(x) \right|^q \right). \\
\end{align*} \]

By substituting (11)-(12) in (9), we easily get the desired result.

**Corollary 2.2.** Since \( \frac{1}{3} < \left( \frac{1}{2p+1} \right)^{\frac{1}{p}} < 1 \), if we choose \( |f''(a)| = |f''(b)| \) in Theorem 2.2, then we obtain:

\[
\left| I(f)(a, b) \right| \leq \left\{ \frac{(x-a)^3 C(p, l_1) + (b-x)^3 C(p, l_2)}{2(b-a)} \right\} L^\frac{1}{q} \left( \left| f''(b) \right|^q, \left| f''(x) \right|^q \right)
\]

**Theorem 2.3.** Let \( f : I \subset R_+ \rightarrow R_+ \) be a twice differentiable function on the interior \( I^0 \) of an interval \( I \) in \( R \) and \( f'' \in L[a, b] \), where \( a, b \in I \) with \( a < b \). If \( |f''|^q \) is geometrically convex and monotonically decreasing on \( [a, b] \) for \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left| I(f)(a, b) \right| \leq \left( \frac{1}{p+1} \right)^\frac{1}{p} \left[ \left( \frac{1}{2} \right) \frac{(x-a)^3}{C(p, l_1) \mu_2^k(q, k_3)} + \frac{(b-x)^3}{2(b-a) C(p, l_1) \mu_2^k(q, k_4)} \right]
\]

where \( C(p, l_i)(i = 1, 2) \) are defined as in Theorem 1.4 and

\[
\begin{align*}
&k_3 = \frac{|f''(a)|^q}{|f''(x)|^q}, \quad k_4 = \frac{|f'(b)|^q}{|f'(x)|^q}, \\
&\mu_2(q, k) = \frac{\Gamma[1 + q, -\ln k] - \Gamma[1 + q]}{\ln k (-\ln k)^q}.
\end{align*}
\]

**Proof.** From Lemma 1 and using Theorem 1.4, we get

\[
\left| I(f)(a, b) \right| \leq \frac{(x-a)^3}{2(b-a) C(p, l_1)} \left( \int_0^1 t^p dt \right)^\frac{1}{p} \left( \int_0^1 t^q |f''(ta + (1-t)x)|^q dt \right)^\frac{1}{q}
\]
\[
+ \frac{(b-x)^3}{2(b-a)} C(p,l_2) \left( \int_0^1 t^p dt \right)^\frac{1}{2} \left( \int_0^1 t^q |f''(tb+(1-t)x)|^q dt \right)^\frac{1}{q} \\
= \left( \frac{1}{p+1} \right)^\frac{1}{p} \left[ \frac{(x-a)^3}{2(b-a)} C(p,l_1) \left( \int_0^1 t^q |f''(ta+(1-t)x)|^q dt \right)^\frac{1}{q} \right] \\
+ \frac{(b-x)^3}{2(b-a)} C(p,l_2) \left( \int_0^1 t^q |f''(tb+(1-t)x)|^q dt \right)^\frac{1}{q}. \quad (13)
\]

Note that, for \( t \in [0,1] \), we have \( a^tb^{1-t} \leq ta + (1-t)b \).

Since \( |f''| \) is geometrically convex and monotonically decreasing on \([a,b]\), for \( t \in [0,1] \) we have that
\[
|f''(ta+(1-t)x)|^q \leq \left| f''(a^tx^{1-t}) \right|^q \leq \left( \left| f''(a) \right|^t \left| f''(x) \right|^{1-t} \right)^q,
\]
\[
|f''(tb+(1-t)x)|^q \leq \left| f''(b^tx^{1-t}) \right|^q \leq \left( \left| f''(b) \right|^t \left| f''(x) \right|^{1-t} \right)^q. \quad (14)
\]

Therefore, by (14) we obtain
\[
(a) \int_0^1 t^q |f''(ta+(1-t)x)|^q dt \\
\leq \int_0^1 t^q |f''(a^tx^{1-t})|^q dt \\
\leq \int_0^1 t^q \left( \left| f''(a) \right|^t \left| f''(x) \right|^{1-t} \right)^q dt \\
= \left| f''(x) \right|^q \mu_2(q,k_3) \quad (15)
\]
and
\[
(b) \int_0^1 t^q |f''(tb+(1-t)x)|^q dt \leq \left| f''(x) \right|^q \mu_2(q,k_4). \quad (16)
\]

By substituting (15)-(16) in (13), we easily get the desired result.

**Corollary 2.3.** Since \( \left( \frac{1}{p+1} \right)^\frac{1}{p} < 1 \) for \( 1 < p < \infty \), if we choose \( |f''(a)| = |f''(b)| \) in Theorem 2.2, then we obtain:
\[
|I(f)(a,b)| \leq \left\{ \frac{(x-a)^3C(p,l_1) + (b-x)^3C(p,l_2)}{2(b-a)} \right\}^{\frac{1}{q}} \mu_2(q,k_3)
\]
Some new Hermite-Hadamard-like type inequalities

**Theorem 2.4.** Let $f : I \subset R_+ \to R_+$ be a twice differentiable function on the interior $I^0$ of an interval $I$ in $R$ and $f'' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f''|^q$ is geometrically convex and monotonically decreasing on $[a, b]$ for $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$\left| I(f)(a, b) \right| \leq \left( \frac{1}{3} \right)^{\frac{1}{p}} \left[ \frac{(x-a)^3}{2(b-a)} \mu_1^\frac{1}{p} \left( \left| f''(a) \right|^q, \left| f''(x) \right|^q, k_3 \right) + \frac{(b-x)^3}{2(b-a)} \mu_1^\frac{1}{p} \left( \left| f''(b) \right|^q, \left| f''(x) \right|^q, k_4 \right) \right]$$

where $\mu_1(\cdot, \cdot, \cdot)$ are defined as in Theorem 2.1 and $k_i(i = 3, 4)$ is defined as in Theorem 2.3.

**Proof.** From Lemma 1 and using the well-known power mean inequality, we get

$$\left| I(f)(a, b) \right| \leq \frac{(x-a)^3}{2(b-a)} \left( \int_0^1 t^2 \left| f''(ta + (1-t)x) \right|^q dt \right)^\frac{1}{q} + \frac{(b-x)^3}{2(b-a)} \left( \int_0^1 t^2 \left| f''(tb + (1-t)x) \right|^q dt \right)^\frac{1}{q}$$

$$= \left( \frac{1}{3} \right)^{\frac{1}{p}} \left[ \frac{(x-a)^3}{2(b-a)} \left( \int_0^1 t^2 \left| f''(ta + (1-t)x) \right|^q dt \right)^\frac{1}{q} + \frac{(b-x)^3}{2(b-a)} \left( \int_0^1 t^2 \left| f''(tb + (1-t)x) \right|^q dt \right)^\frac{1}{q} \right]. \quad (17)$$

Note that, for $t \in [0, 1]$, we have $a^t b^{1-t} \leq ta + (1-t)b$.

Since $|f''|$ is geometrically convex and monotonically decreasing on $[a, b]$, for $t \in [0, 1]$, by using the inequalities (14) we have that

$$(a) \int_0^1 t^2 \left| f''(ta + (1-t)x) \right|^q dt \leq \int_0^1 t^2 \left| f''(a^t x^{1-t}) \right|^q dt$$

$$\leq \int_0^1 t^2 \left| f''(a) \right|^q \left| f''(x) \right|^{q-1} dt = \mu_1 \left( \left| f''(a) \right|^q, \left| f''(x) \right|^q, k_3 \right), \quad (18)$$

$$(b) \int_0^1 t^q \left| f''(tb + (1-t)x) \right|^q dt \leq \mu_1 \left( \left| f''(b) \right|^q, \left| f''(x) \right|^q, k_4 \right). \quad (19)$$

By substituting (19)-(20) in (17), we easily get the desired result.
References


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