On J-Multipliers and J-Multipliers Quadratic of Jordan Banach Algebras

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Abstract
In this work, we generalize the notion of multipliers to Jordan-Banach algebras. We show that a J-multiplier satisfies the condition $(T[U_x(y)]) = U_x[T(y)]$. This suggests that we define a new concept which is the J-Multipliers Quadratic. We show several algebraic or topological properties for both concepts. In particular, we extend some known results for multipliers to J-Multipliers Quadratic.

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1 Introduction
The idea of multipliers has first appeared in Harmonic Analysis with the theory of summability of fourier series[4]. The notion of multipliers has been used in the search of homomorphisms of algebraic groups as well as in the general theory of Banach algebras and in differential equations. The aim of this work is to study the J-multipliers and J-multipliers quadratic in the case of Jordan-Banach algebras. In the first part, we are going to resume the following definition of a multiplier $T$ which is due to Helgason [2]: $T(x,y) = x.Ty$. We will show that some
results that are true in the case of Banach algebras remain true in the case of Jordan-Banach. We will demonstrate that every J-multiplier of Jordan-Banach algebras verifies the relation:

\[ T[U_x(y)] = U_x[T(y)], \forall x, y \in A \]

And that this condition is not sufficient for a linear operator to be a J-multiplier. This suggests to us to define a new notion which is that of J-multiplier quadratic.

In the second part, we will define J-multipliers of Jordan-Banach algebras. We will show that this generalized definition is the one given in the case of Banach. We will compare the two notions just defined and show some theorems of the type Wang. More particularly, we will establish some algebraic and topological relations between all J-multipliers quadratic \( M_{JQ}(A) \) of Jordan-Banach and the algebra of continuous linear operators of \( A \), noted \( L(A) \). We will also show some algebraic properties of \( M_{JQ}(A) \).

## 2 Preliminaries

Let \( \mathbb{K} \) be a commutative field of characteristic zero.

We call \( \mathbb{K} \)-algebra every \( \mathbb{K} \)-vector space \( A \) provided with a bilinear product \( (x, y) \rightarrow x.y \) of \( A \times A \) in \( A \). if the product is associative (resp. commutative), we say that the algebra is associative (resp. commutative).

If \( A \) is any algebra, we note \( A^+ \) the algebra of the same vector space structure as \( A \), provided with product \( \circ \) Defined by.

\[ x \circ y = \frac{1}{2}(xy + yx), \forall x, y \in A \]

\( \circ \) is called Jordan product.

If \( A \) is an associative algebra not commutative, the subalgebras of \( A^+ \) are called special Jordan algebras.

Among the remarkable identities of these algebras, we note the two following identities:

\[ x \circ y = y \circ x \quad (C) \]

\[ (x^2 \circ y) \circ x = x^2 \circ (y \circ x) \quad (J) \]
More generally, an algebra $(A, \cdot)$ that verifies the two previous identities is called Jordan algebra.

In a Jordan algebra $A$, we define the following applications:

$$R_x(y) = yx$$

$$L_x(y) = xy$$

$$U_x(y) = 2x(xy) - x^2y$$

$$U_{xy} = \frac{1}{2}(U_{x+y} - U_x - U_y)$$

The application $x \to U_x$ is quadratic, thus we have:

$U_{ax} \to a^2U_x$ for every $a \in \mathbb{K}$ and the application $(x, y) \to U_{xy}$ is bilinear.

If $A$ has a unity, then $R_x = U_{x,e} = U_{e,x}$

In the case of special Jordan algebra, we have

$$U_x(y) = xyx, (\forall x, y \in A)$$

$a \in A$ is invertible if and only if $U_a$ is invertible in $\mathcal{L}(A)$ and we have

$$U_{a^{-1}} = (U_a)^{-1}$$

**Theorem 2.1 (Macdonald)**

If a polynomial identity an $x$, $y$ and $z$ of degree at most one on $z$ is satisfied in all special Jordan algebra, then it is again true for all Jordan algebra.

A normed Jordan algebra $A$ is a Jordan algebra provided with a vector space norm $\| \cdot \|$ verifying

$$\|xy\| \leq \|x\| \|y\|, (\forall x, y \in A)$$

A Jordan Banach algebra $(A, \| \cdot \|)$ is a normed Jordan algebra and complete for its norm.

The spectrum of an element $x$ of Jordan algebra $A$ is a set of scalars $\lambda$ verifying that $x - \lambda e$ is not invertible in $A$.

The spectrum of an element of Jordan-Banach algebra is a nonempty compact.

### 3 The J-multipliers of Jordan Banach Algebra

In [2], Helgason has defined a multiplier $T$ of Banach algebra by posing for $x$ et $y$ elements of $A$:

$$(Tx).y = x.(Ty)$$
Then he has showed that if $A$ is without order then this definition is equivalent to:

$$T(x.y) = x.Ty$$ (1)

This remark has allowed Wang to establish a number of results on multipliers [6].
As the relation (1) arises out quite simply of the associativity of $A$, we have had the idea of taking it as a definition of a J-multiplier of Jordan Banach algebra in order to find in the case Jordan some results of the type Wang.

**Definition 3.1** Let $A$ be a Jordan-Banach algebra. $A$ is without order if for all $x$ element of $A$, $xA = 0$ implies $x = 0$, or, for all $x$ element of $A$, $Ax = 0$ implies $x = 0$.

Obviously if $A$ has a unit or if $A$ is semi-simple, then it is without order.

**Definition 3.2** Let $A$ be a Jordan-Banach algebra and $T : A \rightarrow A$ an application.
We say that $T$ is a J-multiplier of $A$ if, for all $x$ and $y$ elements of $A$, we have:

$$T(x.y) = x.Ty$$

The set of J-multipliers of $A$ is noted $M_J(A)$.

**Remark 3.1** If $A$ is a Jordan-Banach algebra, then the identity of $A$ noted $I_A$ is an element of $M_J(A)$ and every J-multipliers $T$ verifies:

$$(Tx).y = x.(Ty)$$

**Remark 3.2** Note that no assumptions of linearity of continuity are made in the definition of a J-multiplier. Indeed these properties are, in many instances, consequences of the definition, as is seen from the following theorem. More specifically, the linearity stems from the fact that $A$ is without order and continuity is a consequence of the closed graph theorem.

**Theorem 3.1** Let $A$ be a Jordan-Banach algebra without order. Then, so every element of $M(A)$ is linear and continuous.

**Proposition 3.1** Let $A$ be a non-commutative Banach algebra without order. If $T$ is a multiplier of $A$, then $T$ is a J-multiplier in special Jordan algebra $A^+$. 
Proof.: Let $A$ be a non-commutative Banach algebra without order and $T$ a multiplier of $A$. Then we have:

\[ T(x \circ y) = T\left(\frac{1}{2}(x.y + y.x)\right) = \frac{1}{2}[T(x.y) + T(y.x)] = \frac{1}{2}[x.T(y) + T(y).x] = x \circ T(y) \]

\[ \square \]

Remark 3.3 The definition of a J-multiplier of Jordan Banach algebra generalizes well that given in the case of Banach algebra.

For the adopted definition of a J-multiplier of Jordan Banach algebra, we have the following theorems the demonstration of which is exactly the same in the case of Banach algebras.

Theorem 3.2 Let $A$ be a Jordan-Banach algebra without order. Then the set $M J(A)$ is a closed and commutative subalgebra of Banach algebra of continuous linear operators $L(A)$, for the topology of simple convergence.

Theorem 3.3 Let $A$ be a Jordan-Banach algebra without order. Then $M J(A)$ is complete in the strong operator topology.

Theorem 3.4 Let $A$ be a Jordan-Banach algebra without order. If $T$ is a J-multiplier of $A$, then we have equivalence between the following propositions:

i) $T$ is bijective.

ii) $T^{-1}$ exists and $T^{-1} \in M J(A)$.

Corollary 3.1 Let $A$ be a Jordan-Banach unitary algebra. Then, for every J-multiplier $T$ we have:

\[ Sp_{M J(A)}(A) = Sp_{L(A)}(T) \]

$M J(A)$ is a full subalgebra de $L(A)$.

Theorem 3.5 Let $A$ be a Jordan-Banach algebra. If for every element every $a$ of $A$, the operator $R a$ is a J-multiplier of $A$, then $A$ is associative and the quadratic operator $U a \in M J(A)$. 
Proof.: Let \( a \in A \).
We assume that \( R_a \in M_J(A) \), Then we have:

\[
R_a(x.y) = x.R_a(y), \quad \forall x, y \in A
\]

\[
(x.y).a = x.(y.a), \quad \forall x, y, a \in A
\]

Therefore \( A \) is associative.
As \( U_a = 2(R_a)^2 - R_a^2 \) and \( M_J(A) \) is a subalgebra of \( L(A) \), \( U_a \in M_J(A) \). \( \Box \)

Proposition 3.2 Let \( A \) be a Banach algebra without order and \( T \) a multiplier of \( A \), then we have:

i) \[
U_{T(x)} = T U_x T, \quad \forall x \in A
\]

(2)

ii) (2) remains true in special algebra \( A^+ \).

Proof.: Let \( A \) be a Banach algebra without order and \( T \in M_J(A) \).

i) We have:

\[
U_{T(x)}(y) = (T(x))^2.y, \quad \forall x, y \in A
\]

And:

\[
U_x(T(y)) = x^2.T(y), \quad \forall x, y \in A
\]

Therefore:

\[
T[U_x(T(y))] = T[x^2.T(y)]
\]

\[
= T[x.(x.T(y))]
\]

\[
= T(x).[x.T(y)]
\]

\[
= T(x).[T(x).y]
\]

\[
= [T(x)]^2.y
\]

Therefore we have:

\[
T[U_x(T(y))] = U_{T(x)}(y), \quad \forall x, y \in A
\]

ii) In every special Jordan algebra, we have:

\[
U_x(y) = x.y.x, \quad \forall x, y \in A
\]

Therefore, we have:

\[
U_{T(x)}(y) = T(x).y.T(x)
\]
\[ U_x(T(y)) = x.T(y).x \]

Thus:

\[
T[U_x(T(y))] = T[x.T(y).x] \\
= T[(x.T(y)).x] \\
= [x.T(y)].T(x) \\
= T(x).y.T(x) \\
= U_{T(x)}(y)
\]

Whence:

\[ U_{T(x)} = TU_xT, \quad \forall x \in A \]

Taking this into account, it is normal to ask ourselves the following questions: Given a Jordan-Banach algebra \( A \) and a J-multiplier \( T \), is the (2) relation verified? This is the purpose of the following proposition:

\[ \square \]

**Proposition 3.3** Let \( A \) be a Jordan-Banach algebra. If \( T \) is a J-multiplier of \( A \), we have:

\[ U_{T(x)} = TU_xT, \quad \forall x \in A \]

**Proof.** Let \( T \in M_J(A) \) and \( x, y \in A \).

Then we have:

\[
TU_xT(y) = T[U_x(T(y))] \\
= T[2x.(x.T(y)) - x^2.T(y)] \\
= 2T(x).[x.T(y)] - T(x^2).T(y) \\
= 2T(x).[T(x).y] - [T(x).x].T(y) \\
= 2T(x).[T(x).y] - T(T(x).x).(y) \\
= 2T(x).[T(x).y] - [T(x)]^2.y \\
= U_{T(x)}(y), \quad \forall x, y \in A
\]

Consequently, \( U_{T(x)} = TU_xT \), for all element \( x \) in \( A \). \( \square \)

**Remark 3.4** Generally, the converse is false. This is the case in the following example:

**Exemple 3.1** \( A \) is a Jordan-Banach algebra. For every element \( a \) of \( A \), we have:

\[ U_{U_a(x)} = U_a U_x U_a \]

Therefore \( U_a \) verifies the relation (2) for every \( a \). But the quadratic operator \( U_a \) is not, generally, a multiplier of \( A \).
**Proposition 3.4** Let $A$ be a Jordan-Banach algebra. If $T$ is a J-multiplier of $A$, then for $x$ and $y$ elements of $A$, we have:

$$T[U_x(y)] = U_x[T(y)]$$

**Proof.** Let $T$ be a J-multiplier of $A$, $x$ and $y$ elements of $A$. Then we have:

$$T[U_x(y)] = T[2x.(x.y) - x^2.y] = 2x.T(x.y) - x^2.T(y) = 2x.[x.T(y)] - x^2.T(y) = U_x[T(y)]$$

□

**Remark 3.5** The converse is false. This is the case in the following example:

**Exemple 3.2** Let $A$ be a Jordan-Banach algebra. For every element $a$ of $A$, $R_a$ verifies the relation (3) but $R_a$ is not, generally, a J-multiplier of $A$.

**Proof.** Let $a$, $x$ and $y$ three elements of $A$. Then we have:

$$R_a[U_x(y)] = R_a[2x.(x.y) - x^2.y] = [2x.(x.y) - x^2.y]a$$

$$U_x(R_a(y)) = U_x(y.a) = 2x.[x.(y.a)] - x^2.(y.a)$$

To show that $R_a(U_x(y)) = U_x(R_a(y))$, it is enough, according to the theorem of Macdonald, to verify this equality in every special Jordan algebra. Every special Jordan algebra, we have:

$$U_x(y) = x.y.x$$

$$R_a(z) = z \circ a = \frac{1}{2}(a.z + z.a), \ \forall z \in A$$

So we have:

$$R_a(U_x(y)) = \frac{1}{2}(x.y.x.a + a.x.y.x)$$

$$U_x(R_a(y)) = \frac{1}{2}(x.y.a.x + x.a.y.x)$$

Thus:

$$R_a(U_x(y)) = U_x(R_a(y)), \ \forall x, y \in A$$

□

**Remark 3.6** The previous Remark 3.5., suggests to define a new notion which is that of J-multipliers quadratic; this is what we are doing in the next section.
4 J-Multipliers Quadratic of Jordan Banach Algebra

**Definition 4.1** Let $A$ be Jordan-Banach algebra and $T : A \rightarrow A$ a continuous linear application. We say that $T$ is a J-multiplier quadratic of $A$ if, we have:

$$T[U_x(y)] = U_x[T(y)], \forall x, y \in A$$

The set of a J-multiplier quadratic of $A$ is noted $M_{JQ}(A)$.

**Remark 4.1** The identity of $A$ is an element of $M_{JQ}(A)$.

**Remark 4.2** Unlike the case of multipliers (see previous section), in the definition of J-multiplier quadratic is required that the application is linear and continuous.

**Proposition 4.1** Let $A$ be Banach algebra without order. If $T$ is a multiplier of $A$, then we have:

$$T[U_x(y)] = U_x[T(y)], \forall x, y \in A$$

**Proof.** $T$ is a multiplier of $A$, then we have:

$$T(x.y) = T(x).y = x.T(y), \forall x, y \in A$$

Therefore:

$$T[U_x(y)] = T(x^2.y) = T[x.(x.y)] = x.T(x.y) = x.[x.T(y)] = x^2.T(y) = U_x[T(y)].$$

$\square$

**Proposition 4.2** Let $A$ be Banach algebra without order. If $T$ is a multiplier of $A$, then $T$ is a J-multiplier quadratic in special Jordan algebra $A^+$.

**Proof.** As in $A^+$, $U_x(y) = x.y.x, \forall x, y \in A$, we have:

$$T[U_x(y)] = T(x.y.x) = x.T(y).x = x.T(y).x = U_x[T(y)].$$

$\square$

**Remark 4.3** The definition of J-multiplier quadratic well generalizes that of multiplier given in the case of Banach.

**Remark 4.4** Let $A$ be Jordan-Banach algebra. Then we have $M_J(A) \subset M_{JQ}(A)$ again, this inclusion is strict.
Theorem 4.1 Let $A$ be Jordan-Banach algebra without order. Then $M_{JQ}(A)$ is a closed subalgebra of $\mathcal{L}(A)$, for the topology of simple convergence, which contains the identity.

$\mathcal{L}(A)$ will denote the Banach algebra of all continuous linear operators from $A$ to $A$.

Proof.: Let $T$ and $S$ be two elements of $M_{JQ}(A)$ and $\alpha \in \mathbb{C}$.

For any two elements $x$ and $y$ of $A$, we have:

$$U_x[(\alpha T)(y)] = U_x[\alpha (T(y))] = \alpha U_x(T(y)) = \alpha[T(U_x(y)] = (\alpha T)U_x(y)$$

Therefore, $\alpha T \in M_{JQ}(A)$

$$(TS)[U_x(y)] = T[S(U_x(y)] = T[U_x(S(y))] = U_x[T(S(y))]

Therefore, $TS \in M_{JQ}(A)$

$$U_x[(T + S)(y)] = U_x[T(y) + S(y)] = U_x[T(y)] + U_x[S(y)] = T[U_x(y)] + S[U_x(y)] = (T + S)[U_x(y)]$$

Therefore, $(T + S) \in M_{JQ}(A)$.

Therefore, $M_{JQ}(A)$ is a subalgebra of $\mathcal{L}(A)$ which contains the unit.

Let’s show that $M_{JQ}(A)$ is closed in $\mathcal{L}(A)$ for the topology of simple convergence.

$(T_\lambda)_{\lambda \in \Lambda}$ is a suite of elements of $M_{JQ}(A)$ and $T$ an element de $\mathcal{L}(A)$ such that, for every element $z$ de $A$, we have:

$$\lim_{\lambda \to +\infty} \|T_\lambda(z) - T(z)\| = 0$$

As with any $\lambda \in \Lambda$ and $T_\lambda \in M_{JQ}(A)$, we have:

$$U_x[T_\lambda(y)] = T_\lambda[U_x(y)], \forall x, y \in A$$

Therefore:

$$\|T[U_x(y)] - U_x[T(y)]\| \leq \|T[U_x(y)] - T_\lambda[U_x(y)]\| + \|U_x[T_\lambda(y) - U_x[T(y)]\|$$

Since the application $z \to U_x(z)$ is continuous on $A$ and $(T_\lambda(z))_\lambda$ converges to $T(z)$ for every $z$ in $A$ then:

$(\forall x, y \in A), (\forall \epsilon > 0), (\exists \lambda \in \Lambda)$ Such that:

$$\|T[U_x(y)] - T_\lambda[U_x(y)]\| \leq \epsilon/2$$
\[ \|U_x[T_\lambda(y)] - U_x[T(y)]\| \leq \epsilon/2 \]

Therefore:
\[ \|T[U_x(y)] - U_x[T(y)]\| \leq \epsilon \]

Then we have:
\[ T[U_x(y)] = U_x[T(y)], \forall x, y \in A \]

i.e \( T \in M_{JQ}(A) \).

**Theorem 4.2** Let \( A \) be Jordan-Banach algebra without order.
Then \( M_{JQ}(A) \) is complete in the strong operator topology.

**Proof.** Suppose \((T_\lambda)_{\lambda \in \Lambda}\) is a cauchy net in the strong operator topology. Then for each element \( z \) in \( A \), \((T_\lambda(z))_{\lambda \in \Lambda}\) is a cauchy net in \( A \) that is complete and hence there exists \( T(z) \) in \( A \) such that:
\[
\lim_{\lambda \to +\infty} \|T_\lambda(z) - T(z)\| = 0
\]

For any two elements \( x \) and \( y \) of \( A \), we have:
\[
\|T[U_x(y)] - U_x[T(y)]\| \leq \|T[U_x(y)] - T_\lambda[U_x(y)]\| + \|U_x[T_\lambda(y)] - U_x[T(y)]\|
\]
\[
\|T[U_x(y)] - U_x[T(y)]\| \leq \|(T - T_\lambda)(U_x(y))\| + \|(U_x)[T_\lambda(y) - T(y)]\|
\]

Since \( T_\lambda, T \) and \( z \to U_x(z) \) are linear and continuous applications on \( A \), then:
\[
\|(T - T_\lambda)(U_x(y))\| \leq \|T - T_\lambda\||U_x(y)|
\]

and
\[
\|(U_x)[T_\lambda(y) - T(y)]\| \leq \|U_x\||T - T_\lambda\||y|
\]

Gold was hypothetically:
\[
\lim_{\lambda \to +\infty} \|T_\lambda - T\| = 0
\]

Then
\[
\|T[U_x(y)] - U_x[T(y)]\| = 0
\]

And so
\[ T[U_x(y)] = U_x[T(y)], \forall x, y \in A \]

Therefore \( T \in M_{JQ}(A) \) and \( M_{JQ}(A) \) is complete in the strong operator topology. \( \square \)
Remark 4.5 Unlike the case of multipliers, $M_{JQ}(A)$ algebra is not complete for the topology of simple convergence.

**Theorem 4.3** Let $A$ be Jordan-Banach algebra without order and $T \in M_{JQ}(A)$. Then the following are equivalent:

i) $T$ is bijective.

ii) $T^{-1}$ exists and $T^{-1} \in M_{JQ}(A)$.

**Proof.**

$ii) \Rightarrow i)$ Obvious

$i) \Rightarrow ii)$ $T^{-1}$ is linear. It is continuous according to the theorem of open application.

On the other hand, we have by hypothesis:

$$T[U_x(y)] = U_x[T(y)], \forall x, y \in A$$

Let $x$ and $y$ are two elements of $A$.

We pose $y = T^{-1}(z)$

Then we have:

$$T[U_x(T^{-1}(z))] = U_x[T(T^{-1}(z))] = U_x(z)$$

Thus we obtain:

$$U_x(T^{-1}(z)) = T^{-1}[U_x(z)], \forall x, z \in A$$

Hence $T^{-1}$. $\square$

**Corollary 4.1** Let $A$ be Jordan-Banach algebra without order and $T \in M_{JQ}(A)$. Then we have:

$$Sp_{M_{JQ}(A)}(T) = Sp_{\mathcal{L}(A)}(T)$$

i.e: $M_{JQ}(A)$ is a full subalgebra of $\mathcal{L}(A)$.

**References**


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