Existence of Solution for a Class of Boundary Value Problems

Yong-ming Li

School of Science
Sichuan University of Science and Engineering
Zigong, Sichuan, 643000, China

Abstract

In this paper, we study the existence of solution for the following singular boundary value problem

\[(p(t)u')' = c(t)p(t)f(u), \quad 0 \leq t < \infty, \]
\[u(0) = 0, \quad u(+\infty) = L > 0.\]

where function \(f\) satisfies Lipschitz condition, namely, \(f \in Lip((-\infty, L]).\) We prove that this singular boundary value problem has one strictly solution.

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1 Introduction

The following boundary value problem

\[\gamma \Delta u = \mu(u) - \mu_0, \quad u'(0) = 0, \quad \lim_{t \to \infty} u(t) = u_l > 0, \quad (1.1)\]
has been used to describe the state of fluid in $\mathbb{R}^N$ when the motion of fluid is absent. Here $u$ is the density of the medium, $\mu(u)$ is the chemical potential of a nonhomogeneous fluid, $\gamma$ and $\mu_0$ are suitable constants [1, 2]. Note that $u(0)$ is the density of the gas at the center of the bubble. We are interested in the strictly increasing solution $u$ to the boundary value problem (1.1) with $0 < u(0) < u_l$, since in this case $u$ determines an increasing mass density profile [3]. To find this kind of solution on the half-line $[0, +\infty)$, problem (1.1) has been transformed to

$$(t^{N-1}u'(t))' = 4t^{N-1}x^2(u + 1)u(u - L), \quad u'(0) = 0, \quad u(+\infty) = L > 0,$$

where $N = 2$ or $N = 3$ denote the respective case of plane or spherical bubbles. The alternative form of problem (1.2) arises in nonlinear field theory [4].

In our study, we shall consider a more general situation of problem (1.2), that is,

$$(p(t)u')' = c(t)p(t)f(u), \quad 0 \leq t < \infty, \quad u(0) = 0, \quad u(+\infty) = L > 0.$$

The reason that we are interest in this problem is based on many appeared investigations; for example, for the simplest case $c(t) \equiv 1$, this problem has been investigated in [5, 6, 7] by means of differential and integral inequalities as well as upper and lower function approach; this kind of solution has been discussed in [8, 9] for which $p(t) = t^k, k \in (1, \infty)$ and the shooting argument combining with variational method [10, 11] were successfully utilized. Here it is a pity that we don’t solve the problem (1.3) with the singular boundary value (1.4), but we find one strictly solution of the following equation with the singular boundary value (1.4)

$$(p(t)u')' = c(t)p(t)\tilde{f}(u), \quad 0 \leq t < \infty,$$

where

$$\tilde{f}(x) = \begin{cases} f(M), & x \in (-\infty, M], \\ f(x), & x \in [M, L], \\ 0, & x \in [L, +\infty), \end{cases}$$

$M$ is some constant defined in later essays, and it must mentioned that function $f$ satisfies Lipschitz condition, namely, $f \in Lip((\infty, L])$.

To obtain the main results, we first introduce the following assumptions:

$(H_1)$ $c(t)$ is a bounded and continuous function in $[0, \infty)$ satisfying $c_1 \leq c(t) \leq c_2, t \in [0, \infty)$ for real numbers $c_2 \geq c_1 > 0$;
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(H2) $f(x)$ is a continuous function in $(-\infty, L]$ possessing two zeros with $f(0) = f(L) = 0$, and $xf(x) < 0$ for $x \in (-\infty, L]\setminus\{0\}$, moreover, there exists $L_0 < 0$ such that $F(L_0) > F(L)$, where $F$ is given by

$$F(x) = -\int_0^x f(z)dz, \quad x \in (-\infty, L];$$

(1.7)

(H3) $p(t) \in C([0, \infty)) \cap C^1((0, \infty))$, $p(0) = 0$, $p'(t) > 0$ in $(0, \infty)$;

(H4) there exists $\alpha \in (0, 1)$ such that $p^\alpha(t)/p'(t)$ is bounded as $t \to 0$;

(H5) $p'(t)/p(t)$ is bounded when $t$ is sufficiently large, and there exists a constant $\bar{b} > 0$ such that

$$\int_0^b p(t)dt > F_0 \int_\bar{b}^{b-L_0+L} p(t)dt,$$

where

$$F_0 = \frac{1 + 2c_2F(L)}{2c_1(F(L_0) - F(L))},$$

while $F$ and $c_2$ are given in (1.7) and (H1), respectively.

In addition, we need the following hypothesis

(H6) $\lim_{x \to -\infty} |f(x)|x = 0$.

Remark 1.1 (i) By (H2), there exists a constant $\bar{B} \in (L_0, L)$ such that $F(\bar{B}) = F(L)$.

(ii) It has been shown in [12] that the assumption (H3) together with the simple formula

$$\lim_{t \to \infty} p(t)/p'(t) = +\infty$$

is a sufficient condition for (H5). There are lots of functions with (H3) − (H5) found in [12].

Remark 1.2 We can choose $\varepsilon > 0$ sufficiently small such that $K_1 = c_2\varepsilon(\bar{b} - L_0 + L)^2 < 1$. If (H6) holds, then according to the continuity of $f$, there exists a constant $M' > 0$ such that $|f(x)| \leq M' + \varepsilon|x|$, $\forall x \in (-\infty, L]$. Set

$$K_2 = c_2M'(\bar{b} - L_0 + L)^2 \quad \text{and} \quad M < \min \left\{ L_0, \frac{1}{1 - K_1} \right\}.$$  

(1.9)

2 Main results

We first consider functions $F$ given by (1.7) and $\tilde{F}$ defined by

$$\tilde{F}(u) = \int_0^u \tilde{f}(x + L)dx.$$  

(2.1)
Assumption \((H_2)\) immediately yields the following conclusions.

**Remark 2.1** The function \(F\) is continuous in \((-\infty, L]\), decreasing in \((-\infty, 0)\), increasing in \((0, L]\), \(F(B) > F(L)\) for \(B \in (-\infty, \bar{B})\) and \(F(B) < F(L)\) for \(B \in (\bar{B}, L)\). Moreover, we have

\[
- \int_0^x \tilde{f}(z)dz \begin{cases} > F(M), & x \in (-\infty, M), \\ = F(x), & x \in [M, L], \\ = F(L), & x \in [L, \infty). \\ \end{cases}
\] (2.2)

**Remark 2.2** The function \(\tilde{F}\) is continuous in \((-\infty, +\infty)\) and \(\tilde{F}(x) = F(L) - F(x + L), x \in [L_0 - L, 0]\). Moreover, \(\tilde{F}\) increases on \([L_0 - L, -L]\), decreases on \([-L, 0]\) and \(\tilde{F}(-L) = F(L), \tilde{F}(L_0 - L) = F(L) - F(L_0)\). Furthermore, we have

\[
\tilde{F}(x) \begin{cases} < 0, & x \in (-\infty, \bar{B} - L), \\ > 0, & x \in (\bar{B} - L, 0), \\ = 0, & x \in [0, +\infty) \cup \{\bar{B} - L\}. \\ \end{cases}
\]

We first consider the equation (1.5) with the following initial value

\[
u(0) = B \leq 0, \ u'(0) = 0.\] (2.3)

The solution of which possesses the following properties.

**Lemma 2.1** If \(\tilde{f} \in \text{Lip}((-\infty, +\infty))\) and \((H_1) - (H_3), (H_6)\) and (1.6) are satisfied, then

(i) The problem (1.5) and (2.3) subject to the initial value \(B \in (-\infty, 0]\) possesses a unique solution \(u \in C^1([0, b))) \cap C^2((0, \infty))\) and \(u \equiv 0\) for \(B = 0\).

(ii) Suppose that \(u\) is the solution of problem (1.5) and (2.3) with the initial value \(B \in (B_0 - \delta, B_0 + \delta)\). For each \(b > 0, B_0 \in (-\infty, 0)\) and \(\delta\) small enough such that \((B_0 - \delta, B_0 + \delta) \subseteq (-\infty, 0)\), there exists a constant \(\bar{M} = \bar{M}(b, B_0, \delta) > 0\) such that

\[
|u(t)| + |u'(t)| \leq \bar{M} \text{ for } t \in [0, b], \text{ and } \int_0^b |u'(s)|p'(s)/p(s)ds \leq \bar{M}.\] (2.4)

(iii) Suppose that \(u_i, i = 1, 2\) are unique solutions of problem (1.5) and (2.3) with the initial values \(B = B_i, i = 1, 2\). For each \(b > 0, B_0 < 0\) and \(\epsilon > 0\), there exists \(\delta > 0\) such that for any \(B_1, B_2 \in [B_0, 0)\), if \(|B_1 - B_2| < \delta\), then we have

\[
|u_1(t) - u_2(t)| + |u_1'(t) - u_2'(t)| < \epsilon, \ t \in [0, b].\] (2.5)

**Proof.** It is obvious that \(\tilde{f}\) is Lipschitz and bounded in \((-\infty, +\infty)\) while \((H_1)\) implies the boundness of the function \(c(t)\). Then the proof of (i) is standard and
is similar to that of [6](Lemma 4) by contraction mapping theorem. From the arguments of step 2 and step 3 in [5](Lemma 3), the result (ii) follows immediately. The proof of (iii) is similar to that of [5](Lemma 6) by taking advantage of Gronwall inequality. The proof is complete.

**Remark 2.3** For a given constant $C \in (-\infty, +\infty)$, consider the initial value condition

$$u(a) = C, \ u'(a) = 0, \ a \geq 0. \quad (2.6)$$

The proof of Lemma 2.1(i) shows that the initial value problem (1.5) and (2.6) has a unique solution $u$ in $[a, +\infty)$. Especially, for $C = 0$ or $C \geq L$, we have $u \equiv C$.

**Lemma 2.2** Assume that $(H_1)-(H_3)$ and $(H_5)$ hold. Let $u \in C^1([0, \infty)) \cap C^2((0, \infty))$ be the solution of (1.5). If $u$ increases in $(0, +\infty)$ with $u(t) \in (-\infty, L]$ for $t \in [0, \infty)$, then $\lim_{t \to \infty} u(t)$ equals to 0 or $L$. In addition, we have $\lim_{t \to \infty} u(t) = L$ by further assuming that (1.8) is satisfied, $f'(0)$ exists and is nonzero.

**Proof.** Firstly, we rewrite (1.5) in the equivalent form

$$u''(t) + \frac{p'(t)}{p(t)}u'(t) = c(t)\tilde{f}(u(t)), \quad t \in (0, \infty). \quad (2.7)$$

Notice that the assumption $(H_2)$ and (1.6) enable us to find $t_1 > 0$ such that for $t \geq t_1$, $\tilde{f}(u(t))$ will not change its sign anymore. Then by taking integration on both sides of (2.7) from $t_1$ to $t > t_1$, we have

$$\int_{t_1}^{t} c(s)\tilde{f}(u(s))ds = u'(t) - u'(t_1) + \int_{t_1}^{t} \frac{p'(s)}{p(s)}u'(s)ds, \quad t \geq t_1.$$  

Note that $\lim_{t \to \infty} u'(t) = 0$, and assumptions $(H_3)$ and $(H_5)$ imply $\int_{t_1}^{+\infty} c(s)\tilde{f}(u(s))ds$ converges, which indicates that $c(t)\tilde{f}(u(t)) \to 0$ as $t \to +\infty$. By $(H_1), (H_2)$ and the assumptions on $u$, we find that either $\lim_{t \to \infty} u(t) = 0$ or $\lim_{t \to \infty} u(t) = L$. The rest of the proof is similar to that of [9](Proposition 11) and so we omit it. The proof is complete.

We consider in this section the case that the nonlinear term $f$ satisfies Lipschitz condition, namely, $f \in \text{Lip}((-\infty, L])$. Then according to (1.6), $\tilde{f}$ also satisfies Lipschitz condition in a larger region $(-\infty, +\infty)$. The following result yields three types of the solution for the initial value problem (1.5) and (2.3).

**Proposition 2.3** Assume that $(H_1)-(H_3), (H_5)$ and $(H_6)$ hold and let initial value $B \in (M, 0)$. If $u \in C^1([0, \infty)) \cap C^2((0, \infty))$ is the solution of problem (1.5) and (2.3), then $u$ takes one of the following three types

1. At some $T > 0$, $u(T) = L$ and $u$ strictly increases in $(0, T]$;
2. At some point $\bar{t} > 0$, $u$ either attains a local maximum $u_{\text{max}} \in (0, L)$, and strictly
increases in \((0, \bar{t})\), or strictly increases in \((0, \infty)\) with \(\lim_{t \to \infty} u(t) = 0\); (3) \(u\) strictly increases in \((0, \infty)\) with \(\lim_{t \to \infty} u(t) = L\).

**Proof.** Integrating (1.5) from 0 to \(t\), we have

\[
 u'(t) = \frac{1}{p(t)} \int_0^t s(s)p(s)\tilde{f}(u(s))ds, \quad t > 0. \tag{2.8}
\]

According to this, (1.6) and \((H_1) - (H_3)\), it is clear that \(u\) is strictly increasing for \(t > 0\) when \(u(t) \in (M, 0)\). On the other hand, Remark 2.3 and \((H_2)\) indicate that \(u\) is not a constant in any interval of \((0, \infty)\). These facts combining with Lemma 2.2 immediately lead to the three types of the solution. The proof is complete.

Denote by sets \(I_i(i = 1, 2)\) the disjoint subsets of \((M, 0)\) such that if \(B \in I_i(i = 1, 2)\), the solution of problem (1.5) and (2.3) is type \((i)(i = 1, 2)\). The next result shows that \(I_i(i = 1, 2)\) are open which is essential for us to determine some initial values \(B\), subject to which the solution of problem (1.5) and (2.3) is type (3).

**Theorem 2.4** Assume \(\tilde{f} \in \text{Lip}((\infty, \infty)).\) If \((H_1) - (H_3), (H_5), (H_6)\) hold and \(c_2/c_1 < 1 + F(L)/F(M)\), then \(I_i(i = 1, 2)\) are open.

**Proof.** We divide the proof into two steps in the spirit of [5].

**Step 1.** Let \(B_0 \in I_1\) and denote by \(u_0\) the solution of problem (1.5) and (2.3) subject to the initial value \(B = B_0\). Then \(u_0\) is type (1) given in Proposition 2.3 according to the definition of \(I_1\). Lemma 2.1(iii) indicates that the solution \(u\) of problem (1.5) and (2.3) must be the type (1) if the initial value \(B \in (M, 0)\) is sufficiently close \(B_0\).

Let \(B_0 \in I_2\) and \(u_0\) be the corresponding solution of problem (1.5) and (2.3). Then \(u_0\) is type (2) given in Proposition 2.3. In the first case of type (2), Lemma 2.1(iii) and (2.8) indicate that for initial value \(B\) sufficiently close to \(B_0\), the solution \(u\) of problem (1.5) and (2.3) also attains its first local maximum in \((0, L)\) at some point \(\bar{t}_1 > 0\) and strictly increases in \((0, \bar{t}_1)\).

**Step 2.** To proceed, we consider the second case of type (2) given in Proposition 2.3, i.e., the solution strictly increases in \((0, \infty)\) with its limitation equals to 0. Let \(u_0\) and \(u\) are solutions of problem (1.5) and (2.3) with initial values \(B_0\) and \(B\), respectively. Suppose that \(u_0\) is the second case of type (2) while \(u\) is not when \(B\) sufficiently close to \(B_0\). Then there exists the first zero \(\theta > 0\) (the only one zero) of \(u\). If \(u\) is type (1) in Proposition 2.3, then there exists \(b_0 > 0\) such that \(u(b_0) = L\) and \(u'(b_0) > 0\) according to Remark 2.3. Since \((p(t)u'(t))' = 0\) for \(u(t) > L\), we have

\[
 u'(t) > 0 \quad \text{and} \quad u(t) > L \quad \text{for} \quad t > b_0. \tag{2.9}
\]

If \(u\) is type (3) in Lemma 2.1, then

\[
 \sup\{u(t), t > \max\{b, b_0\}\} = L. \tag{2.10}
\]
Our next task is to show that formulas (2.9) and (2.10) are impossible. Substituting $u_0$ into (2.7), multiplying $u'_0$, and then taking integration from 0 to $t$ in turn lead to
\[
\int_0^t c(s)\tilde{f}(u_0(s))u'_0(s)ds = \frac{u'_0^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)}u'_0^2(s)ds.
\]

By the fact that $\tilde{f}(u_0(s)) \geq 0$, $u'_0(s) \geq 0$, $c_1 \leq c(s) \leq c_2$ for $s \in (0, t)$, and
\[
F(B_0) - F(u_0(t)) = \int_0^t \tilde{f}(u_0(s))u'_0(s)ds,
\]
we have
\[
c_1(F(B_0) - F(u_0(t))) \leq \frac{u'_0^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)}u'_0^2(s)ds \leq c_2(F(B_0) - F(u_0(t))), \quad t > 0.
\]

Let $t \to \infty$ and notice that $u_0(t) \to 0, u'_0(t) \to 0$ as $t \to \infty$, we get
\[
c_1F(B_0) \leq \int_0^\infty \frac{p'(s)}{p(s)}u'_0^2(s)ds \leq c_2F(B_0). \quad (2.11)
\]

We choose $c_0 > 0$ sufficiently small such that
\[
\frac{3c_0 + (c_2 - c_1)F(M)}{c_1} < F(L). \quad (2.12)
\]

This is feasible due to $c_2/c_1 < 1 + F(L)/F(M)$. Then according to (2.11), we can further choose $b > 0$ such that
\[
\int_b^\infty \frac{p'(s)}{p(s)}u'_0^2(s)ds < c_0. \quad (2.13)
\]

Fixed $\delta > 0$ and $\tilde{M} = \tilde{M}(b, B_0, \delta)$ given in Lemma 2.1(ii). Let $\epsilon \in (0, c_0/2\tilde{M})$ and $u$ be the solution of problem (1.5) and (2.3) subject to initial value $B \in (\tilde{M}, 0)$. According to continuous property of $F$ and Lemma 2.1, there exists $\tilde{\delta} \in (0, \delta)$ such that for $|B - B_0| < \tilde{\delta}$ and $t \in [0, b]$, we have
\[
|F(B) - F(B_0)| < c_0/c_1 \text{ and } |u'_0(t) - u'(t)| \leq \epsilon. \quad (2.14)
\]

The second formulas of both (2.14) and (2.4) enable us to make the following estimation with some precisions
\[
\int_0^b \frac{p'(s)}{p(s)}|u'_0^2(s) - u'^2(s)|ds \leq \int_0^b \frac{p'(s)}{p(s)}(|u'_0(s)| + |u'(s)|)|u'_0(t) - u'(t)|ds \leq \epsilon \cdot 2\tilde{M}.
\]
Since $\epsilon \in (0, c_0/2 \bar{M})$, we immediately derive
\[
\int_0^b \frac{p'(s)}{p(s)} |u_0'(s)|^2 ds < c_0.
\] (2.15)

We are now in a position to derive the contradiction to the formulas (2.9) and (2.10). Multiplying (2.7) by $u'$, and then taking integration from 0 to $t$ with $t > \max\{b_0, b\}$ gives
\[
\int_0^t c(s) \tilde{f}(u(s)) u'(s) ds = \frac{u^2(s)}{2} + \int_0^t \frac{p'(s)}{p(s)} u^2(s) ds \geq \int_0^b \frac{p'(s)}{p(s)} u^2(s) ds.
\]

By substituting $u^2(s) = u_0^2(s) - u_0'^2(s) + u_0'^2(s)$ into the last integral and then applying (2.15), we have
\[
\int_0^t c(s) \tilde{f}(u(s)) u'(s) ds > -c_0 + \int_0^b \frac{p'(s)}{p(s)} u_0'^2(s) ds.
\]

We move on to split the last interval $\int_0^b$ into $\int_0^\infty - \int_b^\infty$ and utilize the first inequality of (2.11) and the estimation (2.13). This gives
\[
\int_0^t c(s) \tilde{f}(u(s)) u'(s) ds > -2c_0 + c_1(F(B_0) - F(B)) + c_1 F(B).
\]

Then the first inequality of (2.14) leads to $\int_0^t c(s) \tilde{f}(u(s)) u'(s) ds > -3c_0 + c_1 F(B)$. To proceed, by splitting $\int_0^t$ into $\int_0^\theta + \int_\theta^t$ and noting the fact that
\[
\int_0^\theta c(s) \tilde{f}(u(s)) u'(s) ds + \int_\theta^t c(s) \tilde{f}(u(s)) u'(s) ds \leq c_2 F(B) + c_1 \int_0^{u(t)} \tilde{f}(u(s)) ds,
\]
we derive
\[
-\int_0^{u(t)} \tilde{f}(u(s)) ds < \frac{3c_0 + (c_2 - c_1)F(B)}{c_1}, \quad t > \max\{b_0, b\}.
\]

Since monotonicity of $F$ implies $F(B) < F(M)$, this together with (2.12) leads to
\[
-\int_0^{u(t)} \tilde{f}(u(s)) ds < F(L) \quad \text{for} \quad t > \max\{b_0, b\}.
\]

Therefore, by (2.2) in Remark 2.1, we derive $\sup\{u(t), t > \max\{b, b_0\}\} < L$ which contradicts to (2.9) and (2.10). The proof is complete. \qed

By taking advantage of the fact that $I_i(i = 1, 2)$ are open, we are able to give two existence results referring to the strictly increasing solution of problem (1.5)
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and (1.4) (i.e., Problem (1.3) and (1.4)) under Lipschitz condition.

**Theorem 2.5.** Assume that \( \tilde{f} \in \text{Lip}((\infty, +\infty)) \), \((H_1) - (H_4), (H_6)\) and (1.8) are satisfied. If \( f'(0) < 0 \), then problem (1.5) and (1.4) possesses at least one strictly increasing solution \( u \) with just one zero and \( u(0) \in (M, 0) \).

**Proof.** Firstly, Lemma 2.2 rules out the second case of type (2) given in Proposition 2.3. Secondly, It can be shown that \( I_i (i = 1, 2) \) are nonempty through the same argument for step 1 in the proof of Theorem 2.4. The proof is complete.

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**References**


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