Characterizations of $N^3$ Slant Helices

According to Quaternionic Frame

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Abstract

In this study, we studied some integral characterizations of $N^3$ slant helices according to quaternionic frame in 4-dimensional Euclidean space $E^4$. Furthermore, we obtain some necessary and sufficient conditions for a space curve to be a $N^3$ slant helix according to quaternionic frame.

Mathematics Subject Classification: 53C20, 20G20, 14H45

Keywords: Quaternionic curve, Quaternionic frame, $B_2$ slant helix.

1. Introduction

The quaternions were first defined by Hamilton in 1843. They are actually multi-dimensional complex numbers and imaginary part of a quaternion is an imaginary vector based on three imaginary orthogonal axes.

Some curves are special in differential geometry. They satisfy some relationships between their curvatures and torsions and have an important role. One of these curves is
general helix which is defined by the property that the tangent of the curve makes a constant angle with a fixed straight line called the axis of the general helix. Furthermore, recently new special curves have been defined and studied by Izumiya and Takeuchi. They have defined slant helix which is a special curve whose principal normal vector makes a constant angle with a fixed direction [11]. Kula and et al. studied some characterizations of slant helices in the Euclidean 3-space [7]. After that, Önder and et al. have regarded the notion of slant helix in $E^4$ and defined these new curves as $B_2$-slant helix [10].

Baharathi and Nagaraj defined the quaternionic curves in $E^3$ and $E^4$ and studied the differential geometry of space curves and introduced Frenet frames and formulae by using quaternions [6]. Following, quaternionic inclined curves have been defined and studied by Karadağ and Sivrídağ [9]. In [1], Çöken and Tuna have studied these curves in the semi-Euclidean space $E^2_4$. Quaternionic rectifying curves have been studied by Gungör and Tosun [8]. Gök and et al. have considered the definition given by Önder and et al. for spatial quaternionic curves and have defined quaternionic $B_2$-slant helix. They have given new characterizations for these curves in $E^4$ and $E^2_2$ [4, 5].

In this study, we define quaternionic $B_2$ slant helices, namely $N_3$ slant helices and obtain some necessary and sufficient conditions for a space curve to be a quaternionic $N_3$ slant helix according to quaternionic frame in the Euclidean space $E^4$.

2. Preliminaries

In [3, 12] and [6], a more complete elementary treatment of quaternions and quaternionic curves can be found respectively.

A real quaternion $q$ is defined by

$$q = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$$

(1)

where $a_i$, $(1 \leq i \leq 4)$ are ordinary numbers, and $e_i$, $(1 \leq i \leq 4)$, $e_4 = +1$ are quaternionic units which satisfy the non-commutative multiplication rules

$$\begin{cases} e_i \times e_i = -e_4, \ (1 \leq i \leq 3) \\ e_j \times e_k = -e_i \times e_k = e_i, \ (1 \leq i, j, k \leq 3) \end{cases}$$

(2)

where $(ijk)$ is an even permutation of (123) in the Euclidean space. The algebra of the quaternions is denoted by $Q$ and its natural basis is given by $\{e_1, e_2, e_3, e_4\}$. We can express a real quaternion in (1) by the form

$$q = s_q + v_q,$$

(3)

where $s_q = a_q$ is scalar part and $v_q = a_2e_2 + a_3e_3 + a_4e_4$ is vector part of $q$. So the conjugate of $q = s_q + v_q$ is defined by

$$\bar{q} = s_q - v_q$$

(4)
Using these basic products we can write the symmetric real-valued, non-degenerate, bilinear form as follows:

\[ h : Q \times Q \to IR, \quad (q, p) \mapsto h(q, p) = \frac{1}{2} (q \times p + p \times q) \quad (5) \]

which is called the quaternion inner product [3]. The norm of \( q \) is

\[ \|q\|^2 = h(q, q) = q \times \bar{q} = \bar{q} \times q = a_1^2 + a_2^2 + a_3^2 + a_4^2. \quad (6) \]

If \( \|q\| = 1 \), then \( q \) is called a unit quaternion. Then, the inverse of the quaternion \( q \) is given by

\[ q^{-1} = \frac{\bar{q}}{\|q\|^2}. \quad (7) \]

Let \( q = s_q + v_q = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 \) and \( p = s_p + v_p = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 \) be two quaternions in \( Q \). Then the quaternion product of \( q \) and \( p \) is given by

\[ q \times p = s_q s_p - \langle v_q, v_p \rangle + s_q v_p + s_p v_q + v_q \wedge v_p, \quad (8) \]

where \( \langle \cdot, \cdot \rangle \) and \( \wedge \) denote the inner product and vector product in Euclidean 3-space \( E^3 \), respectively.

And then if \( q + \bar{q} = 0 \), \( q \) is called a spatial quaternion and if \( q - \bar{q} = 0 \), \( q \) is called a temporal quaternion [6].

**Theorem 2.1.** The three-dimensional Euclidean space \( E^3 \) is identified with the space of spatial quaternions \( \{q \in Q : q + \bar{q} = 0\} \) in an obvious manner. Let \( I = [0,1] \) be an interval in real line \( IR \) and let

\[ \alpha : I \subset IR \to Q, \quad s \mapsto \alpha(s) = \sum_{i=1}^3 \alpha_i(s) e_i \quad (9) \]

be an arc-lengthed curve with nonzero curvatures \( \{k, r\} \) and \( \{\hat{t}(s), \hat{n}_i(s), \hat{b}_i(s)\} \) denotes the Frenet frame of the curve \( \alpha(s) \). Then the Frenet formulae of the quaternionic curve \( \alpha(s) \) are given by

\[ \begin{bmatrix} \hat{r}' \\ \hat{n}_1' \\ \hat{n}_2' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & r \\ 0 & -r & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{n}_1 \\ \hat{n}_2 \end{bmatrix} \quad (10) \]

where \( k(s) \) is principal curvature and \( r(s) \) is torsion of \( \alpha(s) \) [6].

**Theorem 2.2.** The four-dimensional Euclidean space \( E^4 \) is identified with the space of unit quaternions. Let \( I = [0,1] \) be an interval in real line \( IR \) and let

\[ \gamma : I \subset IR \to Q, \quad s \mapsto \gamma(s) = \sum_{i=1}^4 \gamma_i(s) e_i, \quad e_4 = +1 \quad (11) \]

be a smooth curve in \( E^4 \) with nonzero curvatures \( \{K, k, r-K\} \) and
\[ T(s), N_1(s), N_2(s), N_3(s) \] denotes the Frenet frame of the curve \( \gamma(s) \). Then the Frenet formulae of the quaternionic curve \( \gamma(s) \) are given by

\[
\begin{bmatrix}
T' \\
N_1' \\
N_2' \\
N_3'
\end{bmatrix} =
\begin{bmatrix}
0 & K & 0 & 0 \\
-K & 0 & k & 0 \\
0 & -k & 0 & r - K \\
0 & 0 & -(r - K) & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N_1 \\
N_2 \\
N_3
\end{bmatrix}
\]

(12)

where \( K(s) \) is principal curvature, \( k(s) \) is torsion and \( (r - K)(s) \) is bitorsion of \( \gamma(s) \) [6].

3. The \( N^3 \) Slant Helices According to Quaternionic Frame in \( E^4 \)

In this section, we give the definition and characterizations of \( N_3 \) slant helices in \( E^4 \) according to quaternionic frame. First, we give the following definition.

**Definition 3.1.** A unit speed quaternionic curve \( \alpha: I \rightarrow E^4 \) is called a \( N_3 \) slant helix if its second binormal unit vector \( N_3 \) makes a constant angle with a fixed direction \( U \).

Then we can give the followings:

**Theorem 3.1.** A regular quaternionic curve \( \alpha \) with \( K > 0, k > 0 \) and \( r - K \neq 0 \) is a quaternionic \( N_3 \) slant helix if and only if the condition

\[
\left( \frac{r - K}{k} \right)^2 + \left[ \frac{1}{K} \left( \frac{r - K}{k} \right) \right]^2 = \tan^2 \theta = \text{const.}
\]

(13)

is satisfied.

**Proof.** If \( \alpha \) be a quaternionic \( N_3 \) slant helix with which \( h(N_3, U) = \cos \theta \), then differentiating this equation and using the Frenet formulas we get \( h(N_2, U) = 0 \). \( U \) is therefore in the subspace \( T - N_1 - N_3 \) and we can written in the form

\[
U = a_1 T + a_2 N_2 + a_3 N_3
\]

(14)

where \( a_1 = \cos \psi, a_2 = \cos \varphi \) and \( a_3 = \cos \theta \) are direction cosines of \( U \) and therefore \( a_1^2 + a_2^2 + a_3^2 = 1 \). The differentiation of (14) gives

\[
(a'_1 - K a_1) T + (a_1 K + a'_2) N_1 + [a_2 k - a_3 (r - K)] N_2 + a'_3 N_3 = 0
\]

and this equation yields
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\[ a_1 = -\frac{1}{K} a_2', \quad a_1' = K a_2 = \frac{K(r - K)}{k} a_3. \]  

(15)

Since

\[ a_2 = \frac{1}{K} a_1' \quad \text{and} \quad a_2' = \left( \frac{1}{K} \right)' a_1' + \frac{1}{K} a_1'' \]

we find the second order linear differential equation in $a_i$ given by

\[ a_i'' - \frac{K'}{K} a_i' + K^2 a_i = 0. \]  

(16)

Letting $t = \int_0^s K(s) ds$ then this differential equation can be rewritten as

\[ a_i'(t) + a_i(t) = 0 \]

The solution of which is

\[ a_i = A \cos \int_0^s K(s) ds + B \sin \int_0^s K(s) ds, \]

where $A$ and $B$ are constants. With the aid of (15) and (17) we have

\[ A \cos \int_0^s K(s) ds + B \sin \int_0^s K(s) ds = -\frac{1}{K} \left( \frac{r - K}{k} \right)' a_3 = a_1, \]

\[ B \cos \int_0^s K(s) ds - A \sin \int_0^s K(s) ds = \frac{r - K}{k} a_3 = a_2. \]

From these equations it follows that the constants $A$ and $B$ are

\[ A = -a_3 \left[ \frac{1}{K} \left( \frac{r - K}{k} \right)' \cos \int_0^s K(s) ds + \frac{r - K}{k} \sin \int_0^s K(s) ds \right] \]

\[ B = a_3 \left[ -\frac{1}{K} \left( \frac{r - K}{k} \right)' \sin \int_0^s K(s) ds + \frac{r - K}{k} \cos \int_0^s K(s) ds \right] \]

hence we get

\[ A^2 + B^2 = \left[ \frac{1}{K} \left( \frac{r - K}{k} \right)' \right]^2 + \left( \frac{r - K}{k} \right)^2 \cos^2 \theta = \sin^2 \theta \]

or

\[ \left( \frac{1}{K} \left( \frac{r - K}{k} \right)' \right)^2 + \left( \frac{r - K}{k} \right)^2 = \tan^2 \theta = \text{const.} \]  

(18)

Conversely, for a regular quaternionic curve this condition is satisfied we can always find a constant vector $U$, which makes a constant angel with the second binormal of the quaternionic curve. Consider the vector defined by
This vector is constant. Indeed the differentiation of $U$, by taking account of the differentiation of (18), gives that $U' = 0$, this means that $U$ is constant vector. The curve is also a quaternionic $N_3$ slant helix.

**Theorem 3.2.** A quaternionic curve $\alpha = \alpha(s)$, in the Euclidean space $E^4$, is a quaternionic $N_3$ slant helix iff the function $f \in C^2$ is such that

$$f(s)K = \left(\frac{r-K}{k}\right)' , \quad f'(s) = -\frac{k(r-K)}{k}. \quad (19)$$

**Proof.** If $\alpha = \alpha(s)$ is a quaternionic $N_3$ slant helix, from Theorem 3.1 we write

$$\frac{r-K}{k} \left(\frac{r-K}{k}\right)' + \frac{1}{K} \left(\frac{r-K}{k}\right)' \left[\frac{1}{K} \left(\frac{r-K}{k}\right)\right]' = 0 \quad (20)$$

and hence

$$\frac{1}{K} \left(\frac{r-K}{k}\right)' = \frac{\frac{-K(r-K)}{k}}{\frac{k}{k} (\frac{r-K}{k})'}. \quad (21)$$

If we have

$$f(s) = -\frac{\frac{-K(r-K)}{k}}{\frac{k}{k} (\frac{r-K}{k})'}, \quad (22)$$

$$f(s)K = \left(\frac{r-K}{k}\right)'$$

is obtained. In addition, from (20) it can be written

$$\left[\frac{1}{K} \left(\frac{r-K}{k}\right)\right]' = -\frac{K(r-K)}{k}. \quad (23)$$

With the aid of (22) and (23) we have

$$f'(s) = -\frac{\frac{K(r-K)}{k}}{k}. \quad (24)$$

Conversely, let $f(s)K = (\frac{-K}{k})'$. If we define $U$ vector by

$$U = \left(-f(s)T + \frac{r-K}{k} N_1 + N_3\right) \cos \theta, \quad (25)$$

because of the second binormal unit vector $N_3$ of $\alpha(s)$ has constant angle $\theta$ with some fixed line $l$ directed in the unit vector $U$; that is $\alpha(s)$ is a quaternionic $N_3$ slant helix.
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**Theorem 3.3.** A quaternionic curve $\alpha = \alpha(s)$, in the Euclidean space $E^4$, is a quaternionic $N_3$ slant helix if

$$\frac{r-K}{k} = A \cos \int_0^s K(s)ds + B \sin \int_0^s K(s)ds. \quad (26)$$

**Proof.** Let $\frac{r-K}{k} = A \cos \int_0^s K(s)ds + B \sin \int_0^s K(s)ds$. From differentiation of (26) we write

$$\left(\frac{r-K}{k}\right)' = K \left[ -A \sin \int_0^s K(s)ds + B \cos \int_0^s K(s)ds \right].$$

If the function defined by

$$f(s) = - A \sin \int_0^s K(s)ds + B \cos \int_0^s K(s)ds,$$

is satisfies condition (19) and (24). This completes the proof of the sufficiency of condition (26).

Suppose that $\alpha(s)$ is a quaternionic $N_3$ slant helix. Then the condition in Theorem 3.2 is satisfied. Let us define a $C^2$ function $\theta$ and $C^1$ functions $g(s)$ and $h(s)$ on $[0, L]$ by

$$\theta(s) = \int_0^s K(s)ds. \quad (27)$$

$$\begin{cases} 
    g(s) = \frac{r-K}{k} \cos \theta - f(s) \sin \theta, \\
    h(s) = \frac{r-K}{k} \sin \theta + f(s) \cos \theta. 
\end{cases} \quad (28)$$

If we differentiate equations (28) with respect to $s$ and take account of (28), (22) and (24), we find that $g'$ and $h'$ are both identically zero. Therefore, $g(s) = A$ and $h(s) = B$, where $A$ and $B$ are constants. Now, substituting these in (28) and solving the resulting equations for $\frac{r-K}{k}$, we get

$$\frac{r-K}{k} = A \cos \theta + B \sin \theta,$$

which is (26).

Finally, to prove the last assertion in Theorem 3.3, for a curve satisfying the condition $\frac{r-K}{k} = A \cos \theta + B \sin \theta$ and the function $f = - A \sin \theta + B \cos \theta$ satisfies condition (22), (24) in Theorem 3.2. Therefore, by Theorem 3.2 this is a quaternionic $N_3$ slant helix.
References


Received: November 11, 2013