Bounded Nonoscillatory Solutions
for a Third Order Difference Equation

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Abstract

The existence results of bounded nonoscillatory solutions for a third order neutral delay difference equation are proved and two nontrivial examples are included in this paper.

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1 Introduction and Preliminaries

The existence of solutions for several classes of linear and nonlinear second and third order difference equations have been studied by many authors, see for example [1]-[14] and the references cited therein. Using the Banach fixed
point theorem, Cheng [4] discussed the existence of a bounded nonoscillatory solution for the second order neutral delay difference equation with positive and negative coefficients

\[ \Delta^2(x_n + px_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad \forall n \geq n_0 \]

under the condition \( p \neq -1 \). Utilizing the Schauder fixed point theorem, Yan and Liu [13] proved the existence of bounded nonoscillatory solutions for the third order difference equation

\[ \Delta^3 x_n + f(n, x_n, x_{n-\tau}) = 0, \quad \forall n \geq n_0. \]

Motivated by the papers mentioned above, in this paper we investigate the following third order neutral delay difference equation

\[ \Delta^3(x_n + ax_{n-\tau}) + \Delta^2 h(n, x_{n-h_1n}, x_{n-h_2n}) + f(n, x_{n-f_1n}, x_{n-f_2n}) = b_n, \quad n \geq n_0, \quad (1.1) \]

where \( a \in \mathbb{R}, \tau \in \mathbb{N}, n_0 \in \mathbb{N}_0, \{b_n\}_{n \in \mathbb{N}_0} \) is a real sequence, \( \bigcup_{l=1}^{2} \{h_{ln}, f_{ln} \}_{n \in \mathbb{N}_0} \subseteq \mathbb{Z} \), and \( h, f : \mathbb{N}_{n_0} \times \mathbb{R}^2 \to \mathbb{R} \) are mappings with

\[ \lim_{n \to \infty} (n - h_{ln}) = \lim_{n \to \infty} (n - f_{ln}) = +\infty, \quad l \in \{1, 2\}. \]

Using the Banach fixed point theorem, we show some existence results of uncountably many bounded nonoscillatory solutions for Eq.(1.1). To illustrate our results, two examples are also included.

Throughout this paper, we assume that \( \Delta \) is the forward difference operator defined by \( \Delta x_n = x_{n+1} - x_n \), \( \mathbb{R} = (-\infty, +\infty) \), \( \mathbb{R}^+ = [0, +\infty) \), \( \mathbb{Z} \) and \( \mathbb{N} \) stand for the sets of all integers and positive integers, respectively,

\[ \mathbb{N}_a = \{n : n \in \mathbb{N} \text{ with } n \geq a\}, \quad \mathbb{Z}_a = \{n : n \in \mathbb{Z} \text{ with } n \geq a\}, \quad a \in \mathbb{Z}, \]

\[ \alpha = \inf\{n - h_{ln}, n - f_{ln} : l \in \{1, 2\}, \, n \in \mathbb{N}_{n_0}\}, \quad \beta = \min\{n_0 - \tau, \alpha\}, \]

\( l^\infty_\beta \) denotes the Banach space of all bounded sequences on \( \mathbb{Z}_\beta \) with norm

\[ \|x\| = \sup_{n \in \mathbb{Z}_\beta} |x_n| \text{ for } x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l^\infty_\beta \]

and

\[ A(N, M) = \{x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l^\infty_\beta : N \leq x_n \leq M, \, n \in \mathbb{Z}_\beta\} \text{ for } M > N > 0. \]

It is easy to see that \( A(N, M) \) is a bounded closed and convex subset of \( l^\infty_\beta \).

By a solution of Eq.(1.1), we mean a sequence \( \{x_n\}_{n \in \mathbb{Z}_\beta} \) with a positive integer \( T \geq n_0 + \tau + |\alpha| \) such that Eq.(1.1) is satisfied for all \( n \geq T \). As is customary, a solution of Eq.(1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory.
2 Uncountably Many Bounded Nonoscillatory Solutions

Now we investigate the existence of uncountably many bounded nonoscillatory solutions for Eq. (1.1).

**Theorem 2.1.** Let $0 \leq |a| < \frac{1}{2}$, $M$ and $N$ be two positive constants with $M(1 - 2|a|) > N$. Assume that there exist four nonnegative sequences $\{P_n\}_{n \in \mathbb{N}_n}$, $\{Q_n\}_{n \in \mathbb{N}_n}$, $\{R_n\}_{n \in \mathbb{N}_n}$, and $\{W_n\}_{n \in \mathbb{N}_n}$ satisfying

\[
|f(n, u_1, u_2) - f(n, \bar{u}_1, \bar{u}_2)| \leq P_n \max\{|u_l - \bar{u}_l| : l \in \{1, 2\}\},
\]

\[
|h(n, u_1, u_2) - h(n, \bar{u}_1, \bar{u}_2)| \leq R_n \max\{|u_l - \bar{u}_l| : l \in \{1, 2\}\},
\]

\[\forall (n, u_1, \bar{u}_1) \in \mathbb{N}_n \times (\mathbb{R}^+ \setminus \{0\})^2, l \in \{1, 2\};\]

\[
|f(n, u_1, u_2)| \leq Q_n \quad \text{and} \quad |h(n, u_1, u_2)| \leq W_n,
\]

\[\forall (n, u_1) \in \mathbb{N}_n \times (\mathbb{R}^+ \setminus \{0\}), l \in \{1, 2\};\]

\[
\sum_{u=n_0}^{\infty} \max\{R_u, W_u\} < +\infty,
\]

\[
\sum_{u=n_0}^{\infty} \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} \max\{P_t, Q_t, |b_t|\} < +\infty.
\]

Then Eq. (1.1) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.

**Proof.** Set $L \in (N + |a|M, M(1 - |a|))$. It follows from (2.3) that there exist $\theta \in (0, 1)$ and $T \geq n_0 + \tau + |a|$ satisfying

\[
\theta = |a| + \sum_{u=T}^{\infty} \left( R_u + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} P_t \right);
\]

\[
\sum_{u=T}^{\infty} \left[ W_u + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} (Q_t + |b_t|) \right] \leq \min\{M(1 - |a|) - L, L - |a|M - N\}.
\]

Define a mapping $S_L : A(N, M) \rightarrow l^\infty_{\beta}$ by

\[
S_L x_n = \begin{cases} 
L - ax_{n-\tau} + \sum_{u=n}^{\infty} \{h(u, x_{u-h_{1u}}, x_{u-h_{2u}}) \\
+ \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} f(t, x_{t-f_{11}}, x_{t-f_{21}}) - b_t \}, & n \geq T, \\
S_L x_T, & \beta \leq n < T
\end{cases}
\]
for \( x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in A(N, M) \). In terms of (2.1), (2.4) and (2.6), we gain that
\[
\|(S_L x)_n - (S_L y)_n\| = |a||x_n - y_n| + \sum_{u=n}^{\infty} \left[ |h(u, x_{u-h1u}, x_{u-h2u}) - h(u, y_{u-h1u}, y_{u-h2u})| + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} |f(t, x_{t-f1t}, x_{t-f2t}) - f(t, y_{t-f1t}, y_{t-f2t})| \right]
\]
\[
\leq |a||x - y|| + \sum_{u=n}^{\infty} \left[ R_u \max\{|x_{u-h1u} - y_{u-h1u}| : l \in \{1, 2\}\} + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} P_t \max\{|x_{t-f1t} - y_{t-f1t}| : l \in \{1, 2\}\} \right]
\]
\[
\leq \left[ |a| + \sum_{u=T}^{\infty} \left( R_u + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} P_t \right) \right] \|x - y\|
\]
\[
= \theta\|x - y\|,
\]
which gives that
\[
\|S_L x - S_L y\| \leq \theta\|x - y\|, \quad x, y \in A(N, M). \tag{2.7}
\]
In view of (2.2), (2.5) and (2.6), we infer that for any \( x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in A(N, M) \) and \( n \geq T \)
\[
\|(S_L x)_n - L\| \leq |a|x_n - y_n| + \sum_{u=n}^{\infty} \left[ h(u, x_{u-h1u}, x_{u-h2u}) + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} [f(t, x_{t-f1t}, x_{t-f2t}) - b_t] \right]
\]
\[
\leq |a||M + \sum_{u=n}^{\infty} \left[ h(u, x_{u-h1u}, x_{u-h2u}) + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} [f(t, x_{t-f1t}, x_{t-f2t}) + |b_t|] \right]
\]
\[
\leq |a||M + \sum_{u=T}^{\infty} \left[ W_u + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} (Q_t + |b_t|) \right]
\]
\[
\leq |a||M + \min\{M(1 - |a|) - L, L - |a|M - N\}
\]
\[
\leq \min\{M - L, L - N\},
\]
which yields that \( S_L(A(N, M)) \subseteq A(N, M) \). Hence (2.7) means that \( S_L \) is a contraction mapping and it has a unique fixed point \( x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in A(N, M) \).
It follows that for $n \geq T + \tau$

$$x_n = L - ax_{n-\tau} + \sum_{u=n}^{\infty} \left\{ h(u, x_{u-h_{1u}}, x_{u-h_{2u}}) \right\} + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} \left[ f(t, x_{t-f_{1t}}, x_{t-f_{2t}}) - b_t \right],$$

which gives that

$$\Delta(x_n + ax_{n-\tau}) = -h(n, x_{n-h_{1n}}, x_{n-h_{2n}}) - \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} \left[ f(t, x_{t-f_{1t}}, x_{t-f_{2t}}) - b_t \right],$$

$$\Delta^2(x_n + ax_{n-\tau}) = -\Delta h(n, x_{n-h_{1n}}, x_{n-h_{2n}}) + \sum_{t=n}^{\infty} \left[ f(t, x_{t-f_{1t}}, x_{t-f_{2t}}) - b_t \right],$$

$$\Delta^3(x_n + ax_{n-\tau}) = -\Delta^2 h(n, x_{n-h_{1n}}, x_{n-h_{2n}}) - f(n, x_{n-f_{1n}}, x_{n-f_{2n}}) + b_n$$

for $n \geq T + \tau$. That is, $x$ is a bounded nonoscillatory solution of Eq.(1.1).

Let $L_1, L_2 \in (N + |a|M, M(1 - |a|))$ and $L_1 \neq L_2$. For each $j \in \{1, 2\}$, we choose a constant $\theta_j \in (0, 1)$, a positive integer $T_j \geq n_0 + \tau + |\alpha|$ and a mapping $S_{L_j}$ satisfying (2.4)-(2.6), where $\theta, L$ and $T$ are replaced by $\theta_j$, $L_j$ and $T_j$, respectively, and

$$\sum_{u=T_3}^{\infty} \left( R_u + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} P_t \right) < \frac{|L_1 - L_2|}{2M} \quad \text{for some } T_3 > \max\{T_1, T_2\}.$$

Obviously, the contraction mappings $S_{L_1}$ and $S_{L_2}$ have the unique fixed points $x = \{x_n\}_{n \in \mathbb{Z}_+}$ and $y = \{y_n\}_{n \in \mathbb{Z}_+} \in A(N, M)$, respectively. That is, $x$ and $y$ are bounded nonoscillatory solutions of Eq.(1.1) in $A(N, M)$. In order to show that Eq.(1.1) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$, we prove only that $x \neq y$. It follows from (2.6) that for $n \geq T_3$

$$x_n = L_1 - ax_{n-\tau} + \sum_{u=n}^{\infty} \left\{ h(u, x_{u-h_{1u}}, x_{u-h_{2u}}) \right\} + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} \left[ f(t, x_{t-f_{1t}}, x_{t-f_{2t}}) - b_t \right],$$

$$y_n = L_2 - ay_{n-\tau} + \sum_{u=n}^{\infty} \left\{ h(u, y_{u-h_{1u}}, y_{u-h_{2u}}) \right\} + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} \left[ f(t, y_{t-f_{1t}}, y_{t-f_{2t}}) - b_t \right],$$
which yield that
\[
|x_n - y_n + ax_{n-\tau} - ay_{n-\tau}| \
\geq |L_1 - L_2| - \sum_{u=n}^\infty \left[ h(u, x_{u-h_{1u}}, x_{u-h_{2u}}) - h(u, y_{u-h_{1u}}, y_{u-h_{2u}}) \right] \
+ \sum_{s=u}^\infty \sum_{t=s}^\infty |f(t, x_{t-f_1t}, x_{t-f_2t}) - f(t, y_{t-f_1t}, y_{t-f_2t})| \
\geq |L_1 - L_2| - \sum_{u=n}^\infty \left( R_u + \sum_{s=u}^\infty \sum_{t=s}^\infty P_t \right) \|x - y\| \
\geq |L_1 - L_2| - 2M \sum_{u=T_3}^\infty \left( R_u + \sum_{s=u}^\infty \sum_{t=s}^\infty P_t \right) \
> 0, \quad n \geq T_3,
\]
which implies that $x \neq y$. This completes the proof.

**Theorem 2.2.** Let $a < -1$, $M$ and $N$ be two positive constants with $M > N$. Assume that there exist two sequences $\{P_n\}_{n \in \mathbb{N}}$ and $\{Q_n\}_{n \in \mathbb{N}}$ satisfying (2.1)–(2.3). Then Eq. (1.1) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.

**Proof.** Put $L \in (M(1+a), N(1+a))$. It follows from (2.3) that there exist $\theta \in (0, 1)$ and $T \geq n_0 + \tau + |\alpha|$ satisfying
\[
\theta = -\frac{1}{a} \left[ 1 + \sum_{u=T}^\infty \left( R_u + \sum_{s=u}^\infty \sum_{t=s}^\infty P_t \right) \right]; (2.8)
\]
\[
\sum_{u=T}^\infty \left[ W_u + \sum_{s=u}^\infty \sum_{t=s}^\infty (Q_t + |b_t|) \right] \leq \min\{L - (1 + a)M, N(1 + a) - L\}. (2.9)
\]
Define a mapping $S_L : A(N, M) \to l^\infty_\beta$ by
\[
S_Lx_n = \begin{cases}
\frac{L}{a} - \frac{x_{n+\tau}}{a} + \frac{1}{a} \sum_{u=n+\tau}^\infty \left\{ h(u, x_{u-h_{1u}}, x_{u-h_{2u}}) \
+ \sum_{s=u}^\infty \sum_{t=s}^\infty f(t, x_{t-f_1t}, x_{t-f_2t}) - b_t \right\}, & n \geq T, \\
S_Lx_T, & \beta \leq n < T
\end{cases}
(2.10)
\]
for $x = \{x_n\}_{n \in \mathbb{Z}_+} \in A(N, M)$. Thus (2.1), (2.8) and (2.10) imply that for
x = \{x_n\}_{n \in \mathbb{Z}, y = \{y_n\}_{n \in \mathbb{Z}} \in A(N, M) \text{ and } n \geq T \}
\begin{align*}
|\langle S_L x \rangle_n - \langle S_L y \rangle_n | & \leq - \frac{\|x - y\|}{a} - \frac{1}{a} \sum_{u = n}^{\infty} \left[ R_u \max \{|x_{u-h_i} - y_{u-h_i}| : l \in \{1, 2\}\} \right] \\
& \quad + \sum_{s = u}^{\infty} \sum_{t = s}^{\infty} P_t \max \{|x_{t-f_i} - y_{t-f_i}| : l \in \{1, 2\}\} \\
& \leq \theta \|x - y\|
\end{align*}

which gives (2.7). By virtue of (2.2), (2.9) and (2.10), we conclude that for any \(x = \{x_n\}_{n \in \mathbb{Z}} \in A(N, M)\) and \(n \geq T\)
\begin{align*}
\langle S_L x \rangle_n & \leq \frac{L}{a} - \frac{M}{a} - \frac{1}{a} \sum_{u = T}^{\infty} \left[ W_u + \sum_{s = u}^{\infty} \sum_{t = s}^{\infty} (Q_t + |b_t|) \right] \\
& \leq \frac{L}{a} - \frac{M}{a} - \frac{1}{a} \min\{L - (1 + a)M, N(1 + a) - L\} \\
& \leq M
\end{align*}

and
\begin{align*}
\langle S_L x \rangle_n & \geq \frac{L}{a} - \frac{N}{a} + \frac{1}{a} \sum_{u = T}^{\infty} \left[ W_u + \sum_{s = u}^{\infty} \sum_{t = s}^{\infty} (Q_t + |b_t|) \right] \\
& \geq \frac{L}{a} - \frac{N}{a} + \frac{1}{a} \min\{L - (1 + a)M, N(1 + a) - L\} \\
& \geq N,
\end{align*}

which imply that \(S_L(A(N, M)) \subseteq A(N, M)\). Thus (2.7) ensures that \(S_L\) is a contraction mapping and hence it has a unique fixed point \(x = \{x_n\}_{n \in \mathbb{Z}} \in A(N, M)\). It is easy to prove that \(x\) is a bounded nonoscillatory solution of Eq.(1.1).

Put \(L_1, L_2 \in (M(1 + a), N(1 + a))\) and \(L_1 \neq L_2\). For each \(j \in \{1, 2\}\), we select a constant \(\theta_j \in (0, 1)\), a positive integer \(T_j \geq n_0 + \tau + |\alpha|\) and a mapping \(S_{L_j}\) satisfying (2.8)-(2.10), where \(\theta, L\) and \(T\) are replaced by \(\theta_j, L_j\) and \(T_j\), respectively, and
\[\sum_{u = T_3}^{\infty} \left( R_u + \sum_{s = u}^{\infty} \sum_{t = s}^{\infty} P_t \right) < \frac{|L_1 - L_2|}{2M}\] for some \(T_3 > \max\{T_1, T_2\}\).

Note that the contraction mappings \(S_{L_1}\) and \(S_{L_2}\) have the unique fixed points \(x = \{x_n\}_{n \in \mathbb{Z}}\) and \(y = \{y_n\}_{n \in \mathbb{Z}} \in A(N, M)\), respectively, and \(x\) and \(y\) are bounded nonoscillatory solutions of Eq.(1.1) in \(A(N, M)\). In order to show
that Eq. (1.1) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$, we need to prove that $x \neq y$. It follows from (2.10) that for $n \geq T_3$

$$x_n = \frac{L_1}{a} - \frac{x_{n+\tau}}{a} + \frac{1}{a} \sum_{u=n+\tau}^{\infty} \left\{ h(u, x_{u-h_{1_{u}}}, x_{u-h_{2_{u}}}) - \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} [f(t, x_{t-f_{1_{t}}}, x_{t-f_{2_{t}}}) - b_t] \right\},$$

$$y_n = \frac{L_2}{a} - \frac{y_{n+\tau}}{a} + \frac{1}{a} \sum_{u=n+\tau}^{\infty} \left\{ h(u, y_{u-h_{1_{u}}}, y_{u-h_{2_{u}}}) - \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} [f(t, y_{t-f_{1_{t}}}, y_{t-f_{2_{t}}}) - b_t] \right\},$$

which mean that for $n \geq T_3$

$$\left| x_n - y_n + \frac{x_{n+\tau} - y_{n+\tau}}{a} \right|$$

$$\geq -\frac{|L_1 - L_2|}{a} + \frac{1}{a} \sum_{u=n+\tau}^{\infty} \left[ R_u \max\{|x_{u-h_{1_{u}}} - y_{u-h_{1_{u}}}| : l \in \{1, 2\}\} + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} P_t \max\{|x_{t-f_{1_{t}}} - y_{t-f_{1_{t}}}| : l \in \{1, 2\}\} \right]$$

$$\geq -\frac{|L_1 - L_2|}{a} + \frac{1}{a} \left( R_u + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} P_t \right) \| x - y \|$$

$$\geq -\frac{|L_1 - L_2|}{a} + \frac{2M}{a} \sum_{u=T_3}^{\infty} \left( R_u + \sum_{s=u}^{\infty} \sum_{t=s}^{\infty} P_t \right)$$

$$> 0,$$

that is, $x \neq y$. This completes the proof. \hfill \Box

The proofs of Theorems 2.3-2.5 below are analogous to that of Theorems 2.1 and 2.2, and hence are omitted.

**Theorem 2.3.** Let $a \in [0, 1)$, $M$ and $N$ are two positive constants with $M(1-a) > N$. Assume that there exist two sequences $\{P_n\}_{n \in \mathbb{N}_0}$ and $\{Q_n\}_{n \in \mathbb{N}_0}$ satisfying (2.1)-(2.3). Then Eq. (1.1) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.

**Theorem 2.4.** Let $a \in (-1, 0]$, $M$ and $N$ be two positive constants with $M > N$. Assume that there exist two sequences $\{P_n\}_{n \in \mathbb{N}_0}$ and $\{Q_n\}_{n \in \mathbb{N}_0}$ satisfying (2.1)-(2.3). Then Eq. (1.1) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$. 
Theorem 2.5. Let \( a > 1, M \) and \( N \) be two positive constants with \( M > N \). Assume that there exist two sequences \( \{P_n\}_{n \in \mathbb{N}_0} \) and \( \{Q_n\}_{n \in \mathbb{N}_0} \) satisfying (2.1)-(2.3). Then Eq.(1.1) possesses uncountably many bounded nonoscillatory solutions in \( A(N, M) \).

3 Examples

Now we construct two examples to explain the results presented in Section 2.

Example 3.1. Consider the third order neutral delay difference equation

\[
\Delta^3 \left( x_n + \frac{1}{4} x_{n-\tau} \right) + \Delta^2 \left( \frac{x_{n-2} - x_{n-3}^2}{n^5 + 2n^2 + 1} \right) + \frac{n^2 + 1}{n^9(1 + x_{n-3}^2 + x_{n-4}^2)} = \frac{(-1)^{n-1}}{n^6(2n-1)}, \quad n \geq 1,
\]

where \( \tau \in \mathbb{N} \) is fixed. Let \( n_0 = 1, a = \frac{1}{4}, \beta = \min\{1 - \tau, -3\}, M \) and \( N \) be two positive constants with \( M > 2N \) and

\[
\begin{align*}
  b_n &= \frac{(-1)^{n-1}}{n^6(2n-1)}, \quad f_1 = 3, \quad f_2 = 4, \quad h_1 = 2, \quad h_2 = 3, \\
  f(n, u, v) &= \frac{n^2 + 1}{n^9(1 + u^2 + v^2)}, \quad h(n, u, v) = \frac{u - v^2}{n^5 + 2n^2 + 1}, \\
  P_n &= \frac{4M(n^2 + 1)}{n^9}, \quad Q_n = \frac{n^2 + 1}{n^9}, \quad R_n = \frac{1 + 2M}{n^5}, \\
  W_n &= \frac{M + M^2}{n^5}, \quad \forall (n, u, v) \in \mathbb{N} \times \mathbb{R}^2.
\end{align*}
\]

It is easy to see that the conditions (2.1)-(2.3) are satisfied. Thus Theorem 2.1 implies that Eq.(3.1) possesses uncountably many bounded nonoscillatory solutions in \( A(N, M) \).

Example 3.2. Consider the third order neutral delay difference equation

\[
\Delta^3 \left( x_n - 2x_{n-\tau} \right) + \Delta^2 \left( \frac{x_{n-3}}{n^8 + x_{n-2}^3} \right) + \frac{(-1)^nnx_{n-6}^3}{n^9 + x_{n-5}^2} = \frac{n^2 - \ln n}{n^{16} + n^5 + 1}, \quad n \geq 2,
\]

where \( \tau \in \mathbb{N} \) is fixed. Let \( n_0 = 2, a = -2, \beta = \min\{2 - \tau, -4\}, M \) and \( N \) be
two positive constants with $M > N$ and

\[
\begin{align*}
    b_n &= \frac{n^2 - \ln n}{n^{16} + n^5 + 1}, & f_1 &= 5, & f_2 &= 6, & h_1 &= 3, & h_2 &= 2, \\
    f(n,u,v) &= \frac{(-1)^n n v^3}{n^9 + u^2}, & h(n,u,v) &= \frac{u}{n^8 + v^4}, \\
    P_n &= \frac{3M^2 n^9 + 5M^4}{n^{17}}, & Q_n &= \frac{M^3}{n^8}, & R_n &= \frac{n^8 + 5M^4}{n^{16}}, \\
    W_n &= \frac{M}{n^8}, & \forall (n, u, v) \in \mathbb{N}_2 \times \mathbb{R}^2.
\end{align*}
\]

Obviously, the conditions (2.1)-(2.3) hold. Therefore Theorem 2.2 guarantees that Eq.(3.2) possesses uncountably many bounded nonoscillatory solutions in $A(N,M)$.

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**References**


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