On Modified Symmetric Single-step Procedure for
Estimating Polynomial Zeros Simultaneously

Mansor Monsi, Nasruddin Hassan* and Syaida Fadhilah M. Rusli

1Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor DE, Malaysia

2School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia 43600 UKM Bangi, Selangor DE, Malaysia

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Abstract

The point symmetric single-step procedure PSS1 has R-order of convergence at least three. This method is modified, and called the midpoint symmetric single-step procedure PMSS1, so that a better rate of convergence is achieved. The convergence analysis of PMSS1 is shown along with its numerical results.

Keywords: midpoint symmetric, R-order of convergence, simple zeros.

1. Introduction

In this paper, we refer to the iterative procedure mentioned in Bakar et. al [1], Monsi et. al [2], Sham et. al [3,4,5] and Jamaluddin et. al [6,7,8,9,10]. The effectiveness of an algorithm is analyzed using the R-order of convergence of the algorithm which is discussed in detail in Ortega and Rheinboldt [11].

Consider $p: \mathcal{C}^1 \rightarrow \mathcal{C}^1$ be a polynomial of degree $n$ defined by
where $a_n = 1$. The equation $p(x) = 0$ can be expressed in the form

$$p(x) = \prod_{j=1}^{n} (x - x_j^*) = 0$$

where $x_j^* (j = 1, \ldots, n)$ are the zeros of the polynomial (1).

2. The Modified Symmetric Single-Step Procedure PMSS1

The point symmetric single-step procedure PSS1 introduced by Monsi [12] is defined as $(i = 1, \ldots, n)$,

\[
x_i^{(k,0)} = x_i^{(k)},
\]

\[
x_i^{(k,1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1} (x_i^{(k)} - x_j^{(k,1)}) \prod_{j=i+1}^{n} (x_i^{(k)} - x_j^{(k,0)})},
\]

\[
x_i^{(k,2)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1} (x_i^{(k)} - x_j^{(k,1)}) \prod_{j=i+1}^{n} (x_i^{(k)} - x_j^{(k,2)})},
\]

\[
x_i^{(k+1)} = x_i^{(k,2)}, \quad (k \geq 0).
\]

We introduce a new modification called PMSS1 as follows.

\[
x_i^{(k,0)} = x_i^{(k)},
\]

\[
x_i^{(k,1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1} (x_i^{(k)} - x_j^{(k,1)}) \prod_{j=i+1}^{n} (x_i^{(k)} - x_j^{(k,0)})},
\]

\[
x_i^{(k,2)} = x_i^{(k)} - \frac{p(x_i^{(k,1)})}{\prod_{j=1}^{i-1} (x_i^{(k,1)} - x_j^{(k,1)}) \prod_{j=i+1}^{n} (x_i^{(k,2)} - x_j^{(k,2)})},
\]

\[
x_i^{(k+1)} = x_i^{(k,2)}, \quad (k \geq 0).
\]

It is to be shown that the corresponding $R$-order of convergence of PMSS1 which is defined by (4) is at least $4$ or $O_R(\text{PMSS1}, x_i^*) \geq 4 (i = 1, \ldots, n)$. 
Theorem

If (i) \( p: \mathbb{C} \to \mathbb{C} \) defined by (1) has \( n \) distinct zeros \( x_i^* \) \((i = 1, \ldots, n)\); (ii) \( |x_i^{(0)} - x_i^*| \leq \theta d / (2n - 1) \) \((i = 1, \ldots, n)\) where \( 0 < \theta < 1 \) and 
\[
d = \min \{ |x_i^* - x_j^*| : i, j = 1, \ldots, n; i \neq j \};
\]
(iii) the sequence \( \{x_i^{(k)}\} \) \((i = 1, \ldots, n)\) are generated from PMSS1 (from (4)), then \( x_i^{(k)} \to x_i^* \) \((k \to \infty)\) and \( O_R(\text{PMSS1}, x_i^*) \geq 4 \) \((i = 1, \ldots, n)\).

Proof

By Lemma 1, Lemma 2 and equations (25) and (27) of Monsi [12] with 
\[
q_i = q_{1,i}, \tilde{x}_i = x_i^{(k)}, \bar{x}_i = x_i^{(k,0)}, \bar{x}_i = x_i^{(k,1)}, \phi_i = \phi_{1,i}(i = 1, \ldots, n),
\]
it follows that for \( i = 1, \ldots, n, k \geq 0, \)
\[
w_i^{(k,1)} = w_i^{(k)} \left\{ \sum_{j=1}^{i-1} \alpha_{ij}^{(k,1)} w_j^{(k,1)} + \sum_{j=i+1}^{n} \alpha_{ij}^{(k,0)} w_j^{(k,0)} \right\},
\]
where 
\[
w_i^{(k,s)} = x_i^{(k,s)} - x_i^*, \quad (s = 0,1,2),
\]
\[
\alpha_{ij}^{(k,1)} = \frac{\prod_{m \neq i,j} (x_j^{(k,1)} - x_m^{*})}{q_{1,i}'(x_j^{(k,1)})(x_j^{(k,1)} - x_i^{(k,1)})} \quad (j = 1, \ldots, i - 1),
\]
and
\[
\alpha_{ij}^{(k,0)} = \frac{\prod_{m \neq i,j} (x_j^{(k,0)} - x_m^{*})}{q_{1,i}'(x_j^{(k,0)})(x_j^{(k,0)} - x_i^{(k)})} \quad (j = i + 1, \ldots, n).
\]
Similarly, by Lemma 1, Lemma 2 and equations (26) and (28) of Monsi [12], with 
\[
q_i = q_{2,i}, \tilde{x}_i = x_i^{(k,1)}, \bar{x}_i = x_i^{(k,1)}, \bar{x}_i = x_i^{(k,2)}, \phi_i = \phi_{2,i} \quad (i = 1, \ldots, n),
\]
it follows that for \( i = 1, \ldots, n, \ k \geq 0, \)
\[
w_i^{(k,2)} = w_i^{(k,1)} \left\{ \sum_{j=1}^{i-1} \beta_{ij}^{(k,1)} w_j^{(k,1)} + \sum_{j=i+1}^{n} \beta_{ij}^{(k,2)} w_j^{(k,2)} \right\},
\]
where
\[
\beta_{ij}^{(k,1)} = \frac{\prod_{m \neq i,j} (x_j^{(k,1)} - x_m^{*})}{q_{2,i}'(x_j^{(k,1)})(x_j^{(k,1)} - x_i^{(k,1)})} \quad (j = 1, \ldots, i - 1),
\]
and
As shown in Monsi [12] that by (6), (7), (9) and (10), we have the following upper bounds for \( \alpha_{ij}^{(k,r)} (r = 0, 1) \) and \( \beta_{ij}^{(k,s)} (s = 1, 2) \):

\[
\alpha_{ij}^{(k,r)} \leq \frac{1}{(n-1)} \left( \frac{2n-1}{d} \right) \quad \text{and} \quad \beta_{ij}^{(k,s)} \leq \frac{1}{(n-1)} \left( \frac{2n-1}{d} \right) \quad (i = 1, ..., n; j \neq i).
\]

It follows from (5)–(7) and Lemma 3 of Monsi [12] that

\[
\left| w_i^{(0,1)} \right| \leq \theta \left| w_i^{(0,0)} \right| \quad (i = 1, ..., n),
\]

and it follows from (8)–(10) and Lemma 3 of Monsi [12] that

\[
\left| w_i^{(0,2)} \right| \leq \theta^3 \left| w_i^{(0,0)} \right|,
\]

whence

\[
\left| w_i^{(1,0)} \right| \leq \theta^3 \left| w_i^{(0,0)} \right| \quad (i = 1, ..., n) \quad \text{follows from (11d)}.
\]

It then follows by induction on \( k \) that \( \forall k \geq 0 \)

\[
\left| w_i^{(k,0)} \right| \leq \theta^{4k-1} \left| w_i^{(0,0)} \right|, \quad \text{whence} \quad x_i^{(k)} \to x_i^* \quad (k \to \infty), \quad (i = 1, ..., n).
\]

Let

\[
h_i^{(k,s)} = \frac{(2n-1)}{d} \left| w_i^{(k,s)} \right| \quad (i = 1, ..., n; s = 0, 1, 2).
\]

Then by (5)–(11), for \( i = 1, ..., n \),

\[
h_i^{(k,1)} \leq \frac{1}{(n-1)} h_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,1)} + \sum_{j=i+1}^{n} h_j^{(k,0)} \right\}, \quad (i = 1, ..., n),
\]

and for \( i = n, ..., 1, \)

\[
h_i^{(k,2)} \leq \frac{1}{(n-1)} h_i^{(k,1)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,1)} + \sum_{j=i+1}^{n} h_j^{(k,2)} \right\}, \quad (i = 1, ..., n).
\]

Let

\[
u_i^{(1,1)} = \begin{cases} 
2 & (i = 1, ..., n-1) \\
3 & (i = n)
\end{cases}
\]

\[
u_i^{(1,2)} = \begin{cases} 
6 & (i = 1) \\
4 & (i = 2, ..., n-1) \\
5 & (i = n)
\end{cases}
\]

and for \( r = 1, 2 \), let
Modified symmetric single-step procedure

\[
u_i^{(k+1,r)} = \begin{cases} 
4u_i^{(k,r)} + 2 & (i = 1) \\
4u_i^{(k,r)} & (i = 2, \ldots, n - 1) \\
4u_i^{(k,r)} + 1 & (i = n)
\end{cases}
\]  \hfill (16)

Then by (14) – (16), for \((\forall k \geq 1)\)

\[
u_i^{(k,1)} = \begin{cases}
\frac{8}{3}(4^{(k-1)}) - \frac{2}{3} & (i = 1) \\
2(4^{(k-1)}) & (i = 2, \ldots, n - 2) \\
\frac{29}{12}(4^{(k-1)}) - \frac{2}{3} & (i = n - 1) \\
\frac{10}{3}(4^{(k-1)}) - \frac{1}{3} & (i = n)
\end{cases}
\]  \hfill (17)

and

\[
u_i^{(k,2)} = \begin{cases}
\frac{20}{3}(4^{(k-1)}) - \frac{2}{3} & (i = 1) \\
\frac{29}{6}(4^{(k-1)}) - \frac{4}{3} & (i = 2) \\
4(4^{(k-1)}) & (i = 3, \ldots, n - 2) \\
\frac{53}{12}(4^{(k-1)}) - \frac{2}{3} & (i = n - 1) \\
\frac{16}{3}(4^{(k-1)}) - \frac{1}{3} & (i = n)
\end{cases}
\]  \hfill (18)

Suppose, without loss of generality, that \(h_i^{(0,0)} \leq h < 1\) \((i = 1, \ldots, n)\).  \hfill (19)

Then by a lengthy inductive argument, it follows from (11) – (18) that for \(i = 1, \ldots, n, k \geq 0\), \(h_i^{(k,1)} \leq h_i^{(k+1,1)}\) and \(h_i^{(k,2)} \leq h_i^{(k+1,2)}\), which results in (18) and (4d), \((\forall k \geq 0)\)

\[
h_i^{(k)} \leq h_i^{k} \quad (i = 1, \ldots, n).
\]  \hfill (20)

By (11) for \(s = 2\),

\[
|w_i^{(k,2)}| = \frac{d}{(2n-1)} h_i^{(k,2)} \quad (i = 1, \ldots, n),
\]
then by (4d), \[ |w_i^{(k+1)}| = \frac{d}{(2n-1)} h_i^{(k+1)} (i = 1, \ldots, n). \]

So \[ |w_i^{(k)}| = \frac{d}{(2n-1)} h_i^{(k)} (i = 1, \ldots, n)(k \geq 0). \quad (21) \]

Let \[ w^{(k)} = \max_{1 \leq i \leq n} \{ |w_i^{(k)}| \}, \quad h^{(k)} = \max_{1 \leq i \leq n} \{ h_i^{(k)} \}. \quad (22) \]

Then, by (20) – (22), \[ w^{(k)} \leq \frac{d}{(2n-1)} h^{4k} \quad (\forall k \geq 0). \] Hence

\[ R_4(w^{(k)}) = \lim_{k \to \infty} \sup \left\{ \left( \frac{w^{(k)}}{h^{4k}} \right)^{1/4} \right\} \leq \lim_{k \to \infty} \sup \left\{ \left( \frac{d}{2n-1} \right)^{1/4} h \right\} = h < 1. \]

Therefore, by the definition of \( R \)-factor in Monsi et. al [2], we have

\[ O_R(\text{PMSS1}, x^*_i) \geq 4 \quad (i = 1, \ldots, n). \]

### 3. Numerical Result and Conclusion

The analysis has clearly shown that the PMSS1 procedure converges faster than the procedure PSS1 of Monsi [12]. We also established an extension of PSS1, namely the point zoro symmetric single-step procedure PZSS1 of Monsi et. al [2], where PZSS1 and PMSS1 have the same rate of convergence.

**Table 1: Number of Iterations and CPU Times**

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Degree</th>
<th>PMSS1</th>
<th>CPU time</th>
</tr>
</thead>
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<td>Number of iterations</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CPU time</td>
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<td>3</td>
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</tr>
<tr>
<td>5</td>
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<td>3</td>
<td>0.203125</td>
</tr>
</tbody>
</table>

Table 1 showed the superiority of PMSS1 over PSS1 in terms of number of iterations and CPU times. The test examples used are from Rusli et. al [13].

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**References**


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