Some Sectional Rate Spaces

K. Chandrasekhara Rao

Department of Mathematics, Srinivasa Ramanujan Centre
SASTRA University, Kumbakonam-612001, India

K. Balasubramanian

Department of Mathematics
SASTRA University, Thanjavur-613401, India

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Abstract

In this paper we find out determining sets for rate spaces $l_\pi$ and $l^2_\pi$.

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Introduction

A sequence whose $k^{th}$ term is $x_k$, is denoted by $x = \{x_k\}$. Let $w = \{\text{all complex sequences}\}$, $l = \{x = \{x_k\} \in w : \sum_{k=1}^{\infty} |x_k| < \infty\}$ and $l_\pi = \{x = \{x_k\} \in w : \sum_{k=1}^{\infty} |x_k|/\pi_k < \infty\}$.

Let $X$ be a linear space over $\mathbb{K}$ (\(\mathbb{R}\) or \(\mathbb{C}\)) and $M \subset X$.
Then $M$ is called absolutely convex set if
$$\forall \alpha, \beta \in \mathbb{K} : (|\alpha| + |\beta| \leq 1 \Rightarrow \alpha M + \beta M \subset M).$$

Let $D = \varnothing \cap B$ where $B$ is the closed unit ball in $X$,
where $\varnothing = \{\text{all finite sequences}\}$.
If $A = D$, then a subset $E$ of $\varnothing$ is called a determining set for $X$, where $A$ is the absolutely convex hull of $E$. 

Section-(1)

Let \( x = \{x_k\} \) be a sequence and \( \{\pi_k\} \) be a sequence of positive terms. We define

(i) \( l_s = \{x: \sum_{k=1}^{\infty} |x_1 + x_2 + \cdots + x_k| < \infty\} \)

(ii) \( l_{s\pi} = \left\{ x: \left\{ \frac{x_k}{\pi_k} \right\} \in l_s \right\} \) where \( x = \{x_k\} \)

(iii) \( x = \{x_k\} \in l_{s\pi} \Rightarrow \|x\|_{s\pi} = \frac{x_1}{\pi_1} + \frac{x_2}{\pi_2} + \frac{x_3}{\pi_3} + \cdots \)

(iv) [Wilansky, 1984] Let \( X \) be a BK-space. Then \( D = D(X) = \{x \in \phi: \|x\| \leq 1\} \). We do not assume that \( X \ni \phi \). That is \( D = \phi \cap (\text{unit closed sphere in } X) \).

Theorem-(1)

Let \( s^{(k)} \) be the sequence \( \left\{0,0,0,0,0,0,0, \ldots, \frac{1}{\pi_k}, \ldots, -1,0,0, \ldots\right\} \) \( (k^{\text{th}} \text{ place}) \)

for each fixed positive integer \( k \). Let \( E = \{s^{(1)}, s^{(2)}, \ldots\} \).

Then \( E \) is a determining set for the space \( l_{s\pi} \), provided

\[
\left| \frac{1}{\pi_k} \right| + \left| \frac{1}{\pi_k} - \frac{1}{\pi_{k+1}} \right| + \left| \frac{1}{\pi_k} - \frac{1}{\pi_{k+1}} \right| + \cdots \leq 1.
\]

Proof

Step 1: Let \( A \) be the absolutely convex hull of \( E \), \( \phi \) be the set of all finite sequences and \( D \) be the set \( \phi \cap B \), where \( B = \text{the closed unit ball of } l_{s\pi} \). That is, \( D = \{x \in l_{s\pi}: \|x\| \leq 1\} \).

Let \( x = \{x_k\} \in A \). Then \( x = \sum_{k=1}^{m} t_k s^{(k)} \) with \( \sum_{k=1}^{m} |t_k| \leq 1 \) \( \ldots \) (1)

\[
\Rightarrow x = t_1 s^{(1)} + t_2 s^{(2)} + \cdots + t_m s^{(m)} = t_1 \left( \frac{1}{\pi_1}, \frac{-1}{\pi_2}, 0, 0, \ldots \right) + t_2 \left( 0, \frac{1}{\pi_2}, \frac{-1}{\pi_3}, 0, 0, \ldots \right) + \cdots + t_m \left( 0, 0, \ldots, \frac{1}{\pi_m}, \frac{-1}{\pi_{m+1}}, 0, 0, \ldots \right)
\]

\[
= \left( \frac{t_1}{\pi_1} - \frac{t_1}{\pi_2}, 0, 0, \ldots \right) + \left( 0, \frac{t_2}{\pi_2}, -\frac{t_2}{\pi_3}, 0, 0, \ldots \right) + \cdots + \left( 0, 0, \ldots, \frac{t_m}{\pi_m}, -\frac{t_m}{\pi_{m+1}}, 0, 0, \ldots \right)
\]
\[ \left( \frac{t_1}{\pi_1}, \frac{t_2-t_1}{\pi_2}, ..., \frac{t_m-t_{m-1}}{\pi_m}, \frac{-t_m}{\pi_{m+1}}, 0, 0, ... \right) \]
\[ \Rightarrow x \in \phi \quad \ldots (2) \]

From \( x = \sum_{k=1}^{m} t_k s^{(k)} \), taking norm on both sides \( \|x\|_{s,\pi} = \left\| \sum_{k=1}^{m} t_k s^{(k)} \right\|_{s,\pi} \). Hence
\[ \|x\|_{s,\pi} \leq \sum_{k=1}^{m} |t_k| \|s^{(k)}\|_{s,\pi} \]
But \( (s^{(k)}) = \left( 0, 0, ..., \frac{1}{\pi_k}, -1, \frac{1}{\pi_{k+1}}, 0, 0, ... \right) \).
\[ \Rightarrow \|s^{(k)}\|_{s,\pi} = |0| + |0| + ... = 0 + 0 + ... + \left| \frac{1}{\pi_k} \right| + 0 + ... + \left| \frac{1}{\pi_k} - \frac{1}{\pi_{k+1}} \right| + ... \]
\[ = \left| \frac{1}{\pi_k} \right| + \left| \frac{1}{\pi_k} - \frac{1}{\pi_{k+1}} \right| + ... \]
\[ \leq 1 \quad \ldots (3) \]
\[ \Rightarrow \|x\|_{s,\pi} \leq \sum_{k=1}^{m} |t_k| \|s^{(k)}\|_{s,\pi} \leq 1 \]
\[ \leq 1 \|s^{(k)}\|_{s,\pi} \text{ by (1).} \]
\[ \leq 1 \text{ by (3).} \]
\[ \Rightarrow x \in B \quad \ldots (4) \]

From (2) and (4), we have \( x \in D \). Thus \( A \subseteq D \quad \ldots (5) \)

**Step 2:** Let \( x = \{x_k\} \in D \). Then \( x \in \phi \) and \( \|x\|_{s,\pi} \leq 1 \quad \ldots (6) \)

Since \( x \in \phi \), we may have \( x = (x_1, x_2, ..., x_m, 0, 0, ...). \) Then
\[ x = t_1 s^{(1)} + t_2 s^{(2)} + ... + t_m s^{(m)} + ... \] where \( t_1 = x_1, \ t_2 = x_1 + x_2, \ ..., t_m = x_1 + x_2 + ... + x_m \).

Hence
\[ |t_1| + |t_2| + ... + |t_m| = |x_1| + |x_1 + x_2| + ... + |x_1 + x_2 + ... + x_m| \]
\[ = \|x\|_{s,\pi} \leq 1, \text{ by (6).} \]

Consequently \( \sum_{k=1}^{m} |t_k| \leq 1 \). Thus \( x = t_1 s^{(1)} + t_2 s^{(2)} + ... + t_m s^{(m)} \) with \( \sum_{k=1}^{m} |t_k| \leq 1 \).
\[ \Rightarrow x \in A \Rightarrow D \subseteq A \quad \ldots (7) \]

From (5) and (7) we get \( A = D \). Hence \( \{s^{(k)}\} \) is a determining set for \( l_{s,\pi} \).
Section-(2)

Definition

Let $l^2 = \{ x \in W : \sum_{k=1}^{\infty} |x_k|^2 < \infty \}$, $l^2_{\pi}$ is the space of all (complex) sequences $x = \{ x_k \}$ such that $\sum_{k=1}^{\infty} \left| \frac{x_k}{\pi_k} \right|^2 < \infty$ with the norm $\|x\|_{l^2_{\pi}} = \left\{ \sum_{k=1}^{\infty} \left| \frac{x_k}{\pi_k} \right|^2 \right\}^{\frac{1}{2}}$. Then with this norm

\[ l^2_{\pi} \text{ is a } BK\text{-space}. \]

Let $\{ l^2 \}_{s_{\pi}}$ denote the space of all those complex sequences $\{ x_k \}$ such that $\{ y_k \} \in l^2_{\pi}$ with $y_k = x_1 + x_2 + \ldots + x_k$ for $k = 1, 2, 3, \ldots$.

In other words, $\{ l^2 \}_{s_{\pi}} = \{ x = \{ x_k \} : \{ y_k \} \in l^2_{\pi} \}$ with the norm

\[ \|x\|_{\frac{l^2_{s_{\pi}}}{2}} = \left\{ \sum_{k=1}^{\infty} \left| \frac{x_1}{\pi_k} + \frac{x_2}{\pi_k} + \ldots + \frac{x_k}{\pi_k} \right|^2 \right\}^{\frac{1}{2}}. \]

For each $k = 1, 2, 3, \ldots$, $s^{(k)}$ denotes the sequence

\[ \left( 0, 0, \ldots, \frac{1}{\pi_k}, \frac{-1}{\pi_{k+1}}, 0, 0, \ldots \right), \frac{1}{\pi_k} \text{ in the } k^{th} \text{ place and } \frac{-1}{\pi_{k+1}} \text{ in the } (k+1)^{th} \text{ place}. \]

Theorem-(2)

For each fixed positive integer $k$, let $s^{(k)}$ be the sequence

\[ \left( 0, 0, \ldots, \frac{1}{\pi_k}, \frac{-1}{\pi_{k+1}}, 0, 0, \ldots \right), \frac{1}{\pi_k} \text{ in the } k^{th} \text{ place}. \text{ Then } E = \{ s^{(1)}, s^{(2)}, \ldots \} \text{ is a determining set for the space } \{ l^2 \}_{s_{\pi}}, \text{ provided } \left| \frac{1}{\pi_k} \right|^2 + \left| \frac{1}{\pi_k} - \frac{1}{\pi_{k+1}} \right|^2 + \ldots \leq 1. \]

Proof

Let $A$ be the absolutely convex hull of $E$ and $\phi$ denote the set of all finite sequences. Let $D = \phi \cap \left( \text{the closed unit ball in } \{ l^2 \}_{s_{\pi}} \right)$ and $x = \{ x_k \} \in A$.

Then $x = \sum_{k=1}^{m} t_k s^{(k)}$ with $\sum_{k=1}^{m} |t_k| \leq 1$

\[ x = t_1 s^{(1)} + t_2 s^{(2)} + \ldots + t_m s^{(m)} \]

\[ = \frac{t_1}{\pi_1} \left( 0, 0, \ldots, \frac{1}{\pi_1}, \frac{-1}{\pi_2}, 0, 0, \ldots \right) + \frac{t_2}{\pi_2} \left( 0, 0, \ldots, \frac{1}{\pi_2}, \frac{-1}{\pi_3}, 0, 0, \ldots \right) + \ldots + \frac{t_m}{\pi_m} \left( 0, 0, \ldots, \frac{1}{\pi_m}, \frac{-1}{\pi_{m+1}}, 0, 0, \ldots \right) \]

\[ = \frac{t_1}{\pi_1} \left( 0, 0, \ldots, \frac{1}{\pi_1}, \frac{-1}{\pi_2}, 0, 0, \ldots \right) + \frac{t_2}{\pi_2} \left( 0, 0, \ldots, \frac{1}{\pi_2}, \frac{-1}{\pi_3}, 0, 0, \ldots \right) + \ldots + \frac{t_m}{\pi_m} \left( 0, 0, \ldots, \frac{1}{\pi_m}, \frac{-1}{\pi_{m+1}}, 0, 0, \ldots \right) \]
Therefore \( x \in \phi \) \( \ldots (9) \)

From \( x = \sum_{k=1}^{m} t_k s^{(k)} \), taking norm on both sides \( \|x\|_{s\pi} = \left\| \sum_{k=1}^{m} t_k s^{(k)} \right\|_{s\pi} \). Hence

\[
\|x\|_{s\pi} \leq \sum_{k=1}^{m} |t_k| \|s^{(k)}\|_{s\pi} \leq \left\| \sum_{k=1}^{m} t_k s^{(k)} \right\|_{s\pi}
\]

\[
\leq |t_1| \|s^{(1)}\| + |t_2| \|s^{(2)}\| + \cdots + |t_m| \|s^{(m)}\|
\]

\[
= |t_1| \|s^{(1)}\| + |t_2| \|s^{(2)}\| + \cdots + |t_m| \|s^{(m)}\|
\]

\[
\leq |t_1| \|s^{(1)}\|^2 + |t_2| \|s^{(2)}\|^2 + \cdots + |t_m| \|s^{(m)}\|^2
\]

But \( (s^{(k)}) = \left(0,0, \ldots, \frac{1}{\pi_k}, -\frac{1}{\pi_{k+1}}, 0, \ldots\right) \). That is \( \|s^{(k)}\|_{s\pi}^2 = \left\| \left(0,0, \ldots, 1, 0\right) \right\|_{s\pi}^2 \).

\[
= \left\{ \left|0\right|^2 + \left|0\right|^2 + \cdots + \left|0\right|^2 + \frac{1}{\pi_k} \right\}^2 + \left\{ \left|0\right|^2 + \left|0\right|^2 + \frac{1}{\pi_k} \right\}^2 + \cdots \leq 1
\]

\[
\|s^{(k)}\|_{s\pi}^2 \leq 1
\]

\[
\|x\|_{s\pi}^2 \leq \sum_{k=1}^{m} |t_k| \|s^{(k)}\|_{s\pi}^2
\]

\[
\|x\|_{s\pi}^2 \leq 1 \text{ by using (10)}.
\]

Thus \( x \in \phi \) with \( \|x\|_{s\pi} \leq 1 \). Therefore \( x \in D \).

Arbitrariness of \( x \in A \) gives \( A \subset D \) \( \ldots (11) \)

On the other hand, let \( x = \{x_k\} \in D \Rightarrow x \in \phi \) and \( \|x\|_{s\pi} \leq 1 \).

Since \( x \in \phi \), we may have \( x = (x_1, x_2, \ldots, x_m, 0,0, \ldots) \).

Then \( x = t_1 s^{(1)} + t_2 s^{(2)} + \cdots + t_m s^{(m)} + \cdots \)

where \( t_1 = \frac{x_1}{\pi_1}, t_2 = \frac{x_1}{\pi_2}, \ldots, t_m = \frac{x_1}{\pi_1} + \frac{x_2}{\pi_2} + \cdots + \frac{x_m}{\pi_m} \).

Hence \( |t_1| + |t_2| + \cdots + |t_m| = \left\{ \frac{x_1^2}{\pi_1} + \frac{x_1^2}{\pi_2} + \cdots + \frac{x_1^2}{\pi_1} + \frac{x_2^2}{\pi_2} + \cdots + \frac{x_m^2}{\pi_2} + \cdots + \frac{x_m^2}{\pi_m} \right\}^{1/2} \).

But \( \|x\|_{s\pi} \leq 1 \). Consequently \( \sum_{k=1}^{m} |t_k| \leq 1 \). Thus \( x = t_1 s^{(1)} + t_2 s^{(2)} + \cdots + t_m s^{(m)} \) with \( \sum_{k=1}^{m} |t_k| \leq 1 \). Therefore \( x \in A \).

Arbitrariness of \( x \) in \( D \) gives \( D \subset A \) \( \ldots (12) \)

From (11) and (12) it follows that \( A = D \). Hence \( E \) is a determining set for \( (l^2)_{s\pi} \).

This completes the proof.
References


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