A Note on the Zeros of Twisted \(q\)-Euler Polynomials with Weak Weight \(\alpha\)

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Abstract

In this paper we observe the behavior of complex roots of twisted \(q\)-Euler numbers and polynomials \(\tilde{E}_{n,q,w}(x)\) with weak weight \(\alpha\), using numerical investigation. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of twisted \(q\)-Euler numbers and polynomials \(\tilde{E}_{n,q,w}(x)\) with weak weight \(\alpha\). Finally, we give a table for the solutions of twisted \(q\)-Euler numbers and polynomials \(\tilde{E}_{n,q,w}(x)\) with weak weight \(\alpha\).

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: Euler numbers and polynomials, twisted \(q\)-Euler numbers and polynomials

1 Introduction

First, we introduce the Euler numbers and Euler polynomials. The Euler numbers \(E_n\) are defined by the generating function:

\[
\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \text{cf. [1, 2, 4, 5] (1.1)}
\]

where we use the technique method notation by replacing \(E^n\) by \(E_n(n \geq 0)\) symbolically. We consider the Euler polynomials \(E_n(x)\) as follows:

\[
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.2}
\]

It is the aim of this paper to observe an interesting phenomenon of ‘scattering’ of the zeros of twisted \(q\)-Euler numbers and polynomials \(\tilde{E}_{n,q,w}(x)\) with
weak weight \(\alpha\). In Section 2, we study twisted \(q\)-Euler numbers and polynomials \(\widetilde{E}^{(\alpha)}_{n,q,w}(x)\) with weak weight \(\alpha\). In Section 3, we describe the beautiful zeros of twisted \(q\)-Euler numbers and polynomials \(\tilde{E}^{(\alpha)}_{n,q,w}(x)\) with weak weight \(\alpha\) using a numerical investigation. We also display distribution and structure of the zeros of twisted \(q\)-Euler numbers and polynomials \(\tilde{E}^{(\alpha)}_{n,q,w}(x)\) with weak weight \(\alpha\) by using computer.

Throughout this paper, we always make use of the following notations: \(\mathbb{N}\) denotes the set of natural numbers, \(\mathbb{Z}\) denotes the set of integers, \(\mathbb{R}\) denotes the set of real numbers and \(\mathbb{C}\) denotes the set of complex numbers. Let \(q\) be a complex number with \(|q| < 1\) and \(\alpha \in \mathbb{Z}\). Now, we define the \(q\)-number of \(x\) as follows:

\[
[x]_q = \frac{1 - q^x}{1 - q}, \text{ (see [1], [3]).}
\]

## 2 The extended twisted \(q\)-Euler numbers and polynomials with weak weight \(\alpha\)

In this section, we introduce twisted \(q\)-Euler numbers \(\widetilde{E}^{(\alpha)}_{n,q,w}\) and polynomials \(E^{(\alpha)}_{n,q,w}(x)\) with weak weight \(\alpha\). Let \(q\) be a complex number with \(|q| < 1\) and \(w\) be the \(p^{N}\)-th root of unity. By the meaning of (1.1) and (1.2), let us define the extended twisted \(q\)-Euler numbers \(\widetilde{E}^{(\alpha)}_{n,q,w}\) and polynomials \(\tilde{E}^{(\alpha)}_{n,q,w}(x)\) with weak weight \(\alpha\) as follows:

\[
\sum_{n=0}^{\infty} \frac{\tilde{E}^{(\alpha)}_{n,q,w}}{n!} t^n = \frac{[2]q^\alpha}{wq^\alpha e^t + 1}. \tag{2.1}
\]

\[
\sum_{n=0}^{\infty} \tilde{E}^{(\alpha)}_{n,q,w}(x) \frac{t^n}{n!} = \left( \frac{[2]q^\alpha}{wq^\alpha e^t + 1} \right) e^{xt}. \tag{2.2}
\]

When \(q \rightarrow 1\), above (2.1) and (2.2) will become the corresponding definitions of the classical Euler numbers \(E_n\) and Euler polynomials \(E_n(x)\). The following elementary properties of the extended twisted \(q\)-Euler numbers \(\tilde{E}^{(\alpha)}_{n,q,w}\) and polynomials \(\tilde{E}^{(\alpha)}_{n,q,w}(x)\) are readily derived from (2.1) and (2.2). We, therefore, choose to omit details involved (see [4]).

**Proposition 2.1** For any positive integer \(n\), we have

\[
\tilde{E}^{(\alpha)}_{n,q,w}(x) = \sum_{k=0}^{n} \binom{n}{k} \tilde{E}^{(\alpha)}_{k,q,w} x^{n-k}.
\]

**Proposition 2.2** For any positive integer \(n\), we have

\[
\tilde{E}^{(\alpha)}_{n,q,w}(x) = (-1)^n w \tilde{E}^{(\alpha)}_{n-1,q,w-1}(1-x).
\]
### 3 Distribution and Structure of the zeros

In this section, we investigate the zeros of twisted $q$-Euler polynomials $\widetilde{E}_{n,q,w}^{(\alpha)}(x)$ by using computer. Let $q$ be a complex number with $0 < q < 1$ and $w = e^{2\pi i}$. We plot the zeros of $\widetilde{E}_{n,q,w}^{(\alpha)}(x), x \in \mathbb{C}$ (Figure 1, Figure 2). In Figure 1(top-left), we choose $n = 20, q = 1/2, w = e^{2\pi i/3}$, and $\alpha = 5$. In Figure 1(top-right), we choose $n = 20, q = 1/1, w = e^{2\pi i}2$ and $\alpha = 5$. In Figure 1(bottom-left), we choose we choose $n = 20, q = 1/2, w = e^{2\pi i}3$ and $\alpha = 5$. In Figure 1(bottom-right), we choose $n = 20, q = 1/2, w = e^{2\pi i}4$ and $\alpha = 5$.

![Figure 1: Zeros of $\widetilde{E}_{n,q,w}^{(\alpha)}(x)$](image)

In Figure 2(top-left), we choose $n = 20, q = 1/2, w = e^{2\pi i}5$, and $\alpha = 10$. In Figure 2(top-right), we choose $n = 20, q = 1/1, w = e^{2\pi i}6$ and $\alpha = 15$. In Figure 2(bottom-left), we choose we choose $n = 20, q = 1/2, w = e^{2\pi i}7$ and $\alpha = 20$. In Figure 2(bottom-right), we choose $n = 20, q = 1/2, w = e^{2\pi i}8$ and $\alpha = 25$. In Figures 1-2, $\widetilde{E}_{n,q,w}^{(\alpha)}(x), x \in \mathbb{C}$, has not $Im(x) = 0$ reflection.
Figure 2: Zeros of $\tilde{E}_{n,q,w}(x)$

symmetry. Our numerical results for numbers of real and complex zeros of $\tilde{E}_{n,q,w}(x)$ are displayed in Table 1.

Table 1. Numbers of real and complex zeros of $\tilde{E}_{n,q,w}(x)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$w = e^{\frac{2\pi i}{2}}$, $\alpha = 10$, $q = \frac{1}{2}$</th>
<th>$w = e^{\frac{2\pi i}{3}}$, $\alpha = 10$, $q = \frac{1}{3}$</th>
</tr>
</thead>
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<tr>
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<td>complex zeros</td>
</tr>
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<td>0</td>
</tr>
<tr>
<td>2</td>
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<td>2</td>
</tr>
<tr>
<td>3</td>
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<td>2</td>
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<tr>
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<td>5</td>
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</tr>
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</tr>
<tr>
<td>8</td>
<td>0</td>
<td>8</td>
</tr>
</tbody>
</table>
A note on the zeros of twisted $q$-Euler polynomials

Next, we calculated an approximate solution satisfying $	ilde{E}_{n,q,w}^{(\alpha)}(x), x \in \mathbb{R}$. The results are given in Table 2.

**Table 2.** Approximate solutions of $	ilde{E}_{n,q,w}^{(\alpha)}(x), q = 1/2, w = e^{\pi i}, \alpha = 10$

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.0009775</td>
</tr>
<tr>
<td>3</td>
<td>-0.090505</td>
</tr>
<tr>
<td>5</td>
<td>-0.20179</td>
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<tr>
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</tr>
<tr>
<td>11</td>
<td>-0.449</td>
</tr>
<tr>
<td>13</td>
<td>-0.53</td>
</tr>
</tbody>
</table>

Plots of real zeros of $	ilde{E}_{n,q,w}^{(\alpha)}(x)$ for $1 \leq n \leq 25, q = 1/2$ structure are presented (Figure 4). In Figure 4(left), we choose $w = e^{2\pi i}$ and $\alpha = 5$. In Figure 4(right), we choose $w = e^{\pi i}$ and $\alpha = 5$. We shall consider the more general open problem. In general, how many roots does $	ilde{E}_{n,q,w}^{(\alpha)}(x)$ have? Prove or disprove: $	ilde{E}_{n,q,w}^{(\alpha)}(x)$ has $n$ distinct solutions. Find the numbers of complex zeros $C_{E_{n,q,w}^{(\alpha)}}(x)$ of $E_{n,q,w}^{(h)}(x), Im(x) \neq 0$. Prove or give a counterexample: Conjecture: Since $n$ is the degree of the polynomial $	ilde{E}_{n,q,w}^{(\alpha)}(x)$, the number of real zeros $R_{E_{n,q,w}^{(\alpha)}}(x)$ lying on the real plane $Im(x) = 0$ is then $R_{E_{n,q,w}^{(\alpha)}}(x) = n - C_{E_{n,q,w}^{(\alpha)}}(x)$, where $C_{E_{n,q,w}^{(h)}}(x)$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_{n,q,w}^{(\alpha)}}(x)$ and $C_{E_{n,q,w}^{(\alpha)}}(x)$. We plot the $	ilde{E}_{n,q,w}^{(\alpha)}(x)$, respectively (Figures 1-4). These figures give mathematicians an unbounded capacity to create visual mathematical in-
vestigations of the behavior of the roots of the twisted $q$-Euler polynomials $\tilde{E}_{n,q}^{(\alpha)}(x)$ with weak weight $\alpha$. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. For related topics the interested reader is referred to [2], [5].

References


Received: October, 2012