Certain Aspects of Univalent Functions
with Negative Coefficients Defined by Rafid Operator

†T.N. Shanmugam, ‡C. Ramachandran and †R. Ambrose Prabhu

†Department of Mathematics
College of Engineering Guindy
Anna University, Chennai 600 025, Tamilnadu, India

‡Department of Mathematics
University College of Engineering Villupuram
Anna University, Villupuram 605 602,Tamilnadu, India

crjsp2004@yahoo.com

Abstract

In our paper, we study a class \( SRA(\lambda, \beta, \alpha, \mu, \theta) \) which consists of analytic and univalent functions with negative coefficients in the open unit disk \( U = \{ z : |z| < 1 \} \) defined by Modified Hadamard product(or Modified convolution) with Rafid-operator, we obtain coefficient bounds and exterior points for this class.Also Distortion Theorem using Fractional Calculus techniques and some results for this class are obtained.

Mathematics Subject Classification: 30C45

Keywords: Univalent function, Hadamard product, Extreme point, Rafid-operator, Distortion Theorem, Fractional Calculus

1 Introduction

Let \( \mathcal{R} \) denote the class of functions of the form :

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n \in \mathbb{N} = \{1, 2, 3, \ldots \}
\]  

(1.1)

which are analytic and univalent in the unit disk \( U = \{ z : |z| < 1 \} \). If \( f \in \mathcal{R} \) is given by (1.1) and \( g \in \mathcal{R} \) is given by,

\[
g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \quad b_n \geq 0
\]

then the modified Hadamard product \( f * g \) of \( f \) and \( g \) is defined by

\[
(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z)
\]  

(1.2)
Recently T.N. Shanmugam and C. Ramachandran [3] have studied the certain sub class of the class \( \mathcal{A} \) for which the Rafid-Operator has some properties.

**Lemma 1.1.** The Rafid-operator of \( f \in \mathcal{R} \) for \( 0 \leq \theta \leq 1, \ 0 \leq \mu \leq 1 \) is denoted by \( R^\theta_\mu \) and defined as following:

\[
R^\theta_\mu(f(z)) = \frac{1}{(1 - \mu)^{(1+\theta)}\Gamma(\theta + 1)} \int_0^\infty t^{\theta-1} e^{-\left(\frac{t}{1-\mu}\right)} f(zt) dt
\]

\[
= z - \sum_{n=2}^\infty K(\theta, \mu, n)a_n z^n
\]

where \( K(\theta, \mu, n) = \frac{(1 - \mu)^{(n-1)}\Gamma(\theta + n)}{\Gamma(\theta + 1)} \)

**Definition 1.1.** A function \( f \in \mathcal{R}, \ z \in \mathbb{U} \) is said to be in the class \( SRA(\lambda, \beta, \alpha, \mu, \theta) \) if and only if the following inequality is satisfied:

\[
Re \left\{ (1 + \frac{zF''(z)}{F'(z)}) - 1 \right\} \geq \beta \left| (1 + \frac{zF''(z)}{F'(z)}) - 2 \right| + \alpha
\]

where \( F(z) = (1 - \lambda)(R^\theta_\mu(f \ast g)(z)) + \lambda z(R^\theta_\mu(f \ast g)(z))' \), \( 0 \leq \alpha < 1, \ 0 \leq \lambda \leq 1, \ \beta \geq 0, \ z \in \mathbb{U}, \ 0 \leq \mu < 1, \ 0 < \theta \leq 1 \) and \( g \in \mathcal{R} \) given by

\[
g(z) = z - \sum_{n=2}^\infty b_n z^n, \quad b_n \geq 0
\]

**Lemma 1.2.** [1] Let \( w = u + iv. \) Then \( Re(w) \geq \sigma \) if and only if \( w - (1 + \sigma) \leq w + (1 - \sigma) \)

**Lemma 1.3.** [1] Let \( w = u + iv \) and \( \sigma \geq 0, \gamma \) is a real number. Then \( Re(w) > \sigma \) \( w - 1 + \gamma \) if and only if \( Re[w(1 + \sigma e^{i\gamma}) - \sigma e^{i\gamma}] > \gamma \).

We aim to study the Coefficient bounds, Extreme points, Application of Fractional calculus and Hadamard product of the class \( SRA(\lambda, \beta, \alpha, \mu, \theta) \).

### 2 Coefficient Bounds and Extreme Points

We obtain the necessary and sufficient condition and extreme points for the functions \( f(z) \) in the class \( SRA(\lambda, \beta, \alpha, \mu, \theta) \)

**Theorem 2.1.** The function \( f(z) \) defined by equation (1.1) is in the class \( SRA(\lambda, \beta, \alpha, \mu, \theta) \) if and only if

\[
\beta + \sum_{n=2}^\infty n(n(1 + \beta) - (\alpha + 2\beta))(1 - \lambda + n\lambda)K(\theta, \mu, n)a_n b_n \leq 1 - \alpha
\]

where \( 0 \leq \alpha < 1, \ 0 \leq \lambda \leq 1, \ \beta \geq 0, \ z \in \mathbb{U}, \ 0 \leq \mu < 1, \ 0 < \theta \leq 1 \)
Proof. From the definition, we have

\[ \Re \left\{ \frac{(R^\theta_\mu(f * g)(z))' + (1 + 2\lambda)z(R^\theta_\mu(f * g)(z))'' + \lambda z^2(R^\theta_\mu(f * g)(z))'''}{(R^\theta_\mu(f * g)(z))' + \lambda z(R^\theta_\mu(f * g)(z))'')} - 1 \right\} \geq \beta \]

\[ \Re \left\{ \frac{(R^\theta_\mu(f * g)(z))' + (1 + 2\lambda)z(R^\theta_\mu(f * g)(z))'' + \lambda z^2(R^\theta_\mu(f * g)(z))'''}{(R^\theta_\mu(f * g)(z))' + \lambda z(R^\theta_\mu(f * g)(z))'')} - 2 \right\} + \alpha \]

From Lemma 1.3, we have

\[ \Re \left\{ \frac{(R^\theta_\mu(f * g)(z))' + (1 + 2\lambda)z(R^\theta_\mu(f * g)(z))'' + \lambda z^2(R^\theta_\mu(f * g)(z))'''}{(R^\theta_\mu(f * g)(z))' + \lambda z(R^\theta_\mu(f * g)(z))'')} (1 + \beta e^{i\phi}) - 2\beta e^{i\phi} \right\} \geq \alpha \]

\[-\pi \leq \phi \leq \pi, \text{ or equivalently,} \]

\[ \Re \left\{ \frac{[(R^\theta_\mu(f * g)(z))' + (1 + 2\lambda)z(R^\theta_\mu(f * g)(z))'' + \lambda z^2(R^\theta_\mu(f * g)(z))'''](1 + \beta e^{i\phi})}{(R^\theta_\mu(f * g)(z))' + \lambda z(R^\theta_\mu(f * g)(z))'') - 2\beta e^{i\phi} \left[ (R^\theta_\mu(f * g)(z))' + \lambda z(R^\theta_\mu(f * g)(z))'' \right]} \right\} \geq \alpha \]

(2.2)

Let

\[ F(z) = \left[ R^\theta_\mu(f * g)(z))' + (1 + 2\lambda)z(R^\theta_\mu(f * g)(z))'' + \lambda z^2(R^\theta_\mu(f * g)(z))''' \right](1 + \beta e^{i\phi}) \]

\[ -2\beta e^{i\phi} \left[ (R^\theta_\mu(f * g)(z))' + \lambda z(R^\theta_\mu(f * g)(z))'' \right] \]

and \( E(z) = (R^\theta_\mu(f * g)(z))' + \lambda z(R^\theta_\mu(f * g)(z))'' \)

By lemma 1.2, equation(2.2) is equivalent to \( |F(z) + (1 - \alpha)E(z)| \geq |F(z) - (1 + \alpha)E(z)| \), for \( 0 \leq \alpha < 1 \) But

\[ \left| F(Z) + (1 - \alpha)E(Z) \right| = \left| 1 - \sum_{n=2}^{\infty} n^2(1 + n\lambda - \lambda)K(n, \mu, \theta) a_n b_n z^{n-1} \right| \]

\[ -\beta e^{i\phi} \left[ 1 - \sum_{n=2}^{\infty} n(2 - n)(1 + n\lambda - \lambda)K(n, \mu, \theta) a_n b_n z^{n-1} \right] \]

\[ +(1 - \alpha) \left[ (1 - \sum_{n=2}^{\infty} n^2(1 + n\lambda - \lambda)K(n, \mu, \theta) a_n b_n z^{n-1}) \right] \]

\[ \geq (2 - \alpha) - \sum_{n=2}^{\infty} n(n + 1 - \alpha)(1 + n\lambda - \lambda)K(n, \mu, \theta) a_n b_n \left| z \right|^{n-1} \]

\[ -\beta \left[ 1 - \sum_{n=2}^{\infty} n(2 - n)(1 + n\lambda - \lambda)K(n, \mu, \theta) a_n b_n \left| z \right|^{n-1} \right] \]
Now,
\[ |F(Z) - (1 + \alpha)E(Z)| = \left| 1 - \sum_{n=2}^{\infty} n^2(1 + n\lambda - \lambda)K(n, \mu, \theta)a_n b_n | z |^{n-1} \right. \]
\[ - \beta \exp^i\phi[1 - \sum_{n=2}^{\infty} n(2 - n)(1 + n\lambda - \lambda)K(n, \mu, \theta)a_n b_n | z |^{n-1}] \]
\[ - (1 + \alpha)(1 - \sum_{n=2}^{\infty} n(1 + n\lambda - \lambda)K(n, \mu, \theta)a_n b_n z^n) \]
\[ \leq \alpha + \sum_{n=2}^{\infty} n(n - 1 - \alpha)(1 + n\lambda - \lambda)K(n, \mu, \theta)a_n b_n | z |^{n-1} \]
\[ + \beta[1 - \sum_{n=2}^{\infty} n(2 - n)(1 + n\lambda - \lambda)K(n, \mu, \theta)a_n b_n | z |^{n-1}] \]

Hence,
\[ |F(z) + (1 - \alpha)E(z)| - |F(z) - (1 + \alpha)E(z)| \geq (2(1 - \alpha)) | z |^{n-1} \]
\[ - 2 \sum_{n=2}^{\infty} n(n - \alpha)(1 + n\lambda - \lambda)K(n, \mu, \theta)a_n b_n | z |^{n-1} \]
\[ - 2\beta[1 + \sum_{n=2}^{\infty} n(n - 2)(1 + n\lambda - \lambda)K(n, \mu, \theta)a_n b_n | z |^{n-1}] \]
\[ \geq 0 \]

or
\[ \sum_{n=2}^{\infty} n(n - \alpha)(1 + n\lambda - \lambda)K(n, \mu, \theta)a_n b_n + \beta[1 + \sum_{n=2}^{\infty} n(n - 2)(1 + n\lambda - \lambda)K(n, \mu, \theta)a_n b_n] \leq 1 - \alpha \]

which is equivalent to
\[ \sum_{n=2}^{\infty} [(1 - \lambda + n\lambda)[n(n - \alpha) + \beta(n(n - 2))] + \beta]K(n, \mu, \theta)a_n b_n \leq 1 - \alpha \]

Conversely suppose that the equation(2.1) holds good, then we have to prove that
\[ \text{Re} \left\{ 1 + \frac{zF''(z)}{F'(z)}(1 + \beta e^{i\phi}) - 2\beta e^{i\phi} \right\} \geq \alpha \]

Now choosing the value of z on the positive real axis where \( 0 \leq |z| = r < 1 \), the above inequality reduces to
\[ \text{Re} \left\{ (1 - \alpha) - \sum_{n=2}^{\infty} n^2(1 + \beta e^{i\phi})(1 - \lambda + n\lambda) - n(\alpha + 2\beta e^{i\phi})(1 - \lambda + n\lambda)]K(n, \mu, \theta)a_n b_n r^{n-2} + \beta e^{i\phi} \right\} \geq 0 \]
Since \( Re(e^{-i\phi}) \geq -|e^{i\phi}| = -1 \), the above inequality reduces to

\[
Re \left\{ \frac{(1 - \alpha) - \sum_{n=2}^{\infty} \left[ n^2(1 + \beta)(1 - \lambda + n\lambda) - n(\alpha + 2\beta)(1 - \lambda + n\lambda) \right] K(n, \mu, \theta) a_n b_n r^{n-2} + \beta}{1 - \sum_{n=2}^{\infty} (1 - \lambda + n\lambda) K(n, \mu, \theta) a_n b_n r^{n-2}} \right\} \geq 0
\]

Letting \( r \to 1^- \), we get the desired result. Hence the proof.

Corollary 2.1. If \( f \in \mathcal{SRA}(\lambda, \beta, \alpha, \mu, \theta) \), then

\[
a_n \leq \frac{1 - \alpha}{\beta + [n(1 + n\lambda - \lambda)(n(1 + \beta) - (\alpha + 2\beta))] K(n, \mu, \theta) b_n},
\]

where \( 0 \leq \alpha < 1, \beta \geq 0, 0 \leq \lambda \leq 1, 0 \leq \mu < 1, 0 < \theta \leq 1 \)

Theorem 2.2. If \( f_1(z) = z \) and

\[
f_n(z) = z - \frac{1 - \alpha}{\beta + [n(1 + n\lambda - \lambda)(n(1 + \beta) - (\alpha + 2\beta))] K(n, \mu, \theta) b_n} z^{n-1},
\]

where \( n \geq 2, n \in \mathbb{N}, 0 \leq \alpha < 1, \beta \geq 0, 0 \leq \lambda \leq 1, 0 \leq \mu < 1, 0 < \theta \leq 1 \). Then \( f \in \mathcal{SRA}(\lambda, \beta, \alpha, \mu, \theta) \) if and only if it can be expressed in the form

\[
f(z) = \sum_{n=2}^{\infty} \sigma_n f_n(z),
\]

where \( \sigma_n \geq 0 \) and \( \sum_{n=1}^{\infty} \sigma_n = 1 \) or \( 1 = \sigma_1 + \sum_{n=2}^{\infty} \sigma_n \).

Proof. Let \( f(z) = \sum_{n=1}^{\infty} \sigma_n f_n(z) \), where \( \sigma_n \geq 0 \) and \( \sum_{n=1}^{\infty} \sigma_n = 1 \) or \( 1 = \sigma_1 + \sum_{n=2}^{\infty} \sigma_n \). Then

\[
f(z) = z - \frac{1 - \alpha}{\beta + [n(1 + n\lambda - \lambda)(n(1 + \beta) - (\alpha + 2\beta))] K(n, \mu, \theta) b_n} z^{n-1},
\]

But

\[
f(z) = \sum_{n=2}^{\infty} \left\{ \frac{\beta + [n(1 + n\lambda - \lambda)(n(1 + \beta) - (\alpha + 2\beta))] K(n, \mu, \theta) b_n}{1 - \alpha} \right\} \sigma_n z^{n-1},
\]

\[
= \sum_{n=2}^{\infty} \sigma_n
\]

\[
= 1 - \sigma_1 \leq 1 (\text{from theorem 2.1})
\]
Using Theorem 2.1, we have \( f \in SRA(\lambda, \beta, \alpha, \mu, \theta) \). Conversely, let us assume that \( f(z) \) of the form (1.1) belongs to \( SRA(\lambda, \beta, \alpha, \mu, \theta) \). Then

\[
a_n \leq \frac{1 - \alpha}{\beta + [n(1 + n\lambda - \lambda)(1 + \beta) - (\alpha + 2\beta)]} K(n, \mu, \theta) b_n, \quad n \in \mathbb{N}, n \geq 2
\]

Setting

\[
\sigma_n = \frac{\beta + [n(1 + n\lambda - \lambda)(1 + \beta) - (\alpha + 2\beta)]} {1 - \alpha} K(n, \mu, \theta) a_n b_n
\]

and

\[
\sigma_1 = 1 - \sum_{n=2}^{\infty} \sigma_n
\]

we have

\[
f(z) = \sum_{n=1}^{\infty} \sigma_n f_n(z)
\]

\[
= \sigma_1 f_1(z) + \sum_{n=2}^{\infty} \sigma_n f_n(z)
\]

Hence the proof.

3 Hadamard Product

**Theorem 3.1.** Let

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n
\]

and

\[
g(z) = z - \sum_{n=2}^{\infty} b_n z^n
\]

belongs to \( SRA(\lambda, \beta, \alpha, \mu, \theta) \). Then the Hadamard Product of \( f(z) \) and \( g(z) \) given by

\[
(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n \text{ belongs to } SRA(\lambda, \beta, \alpha, \mu, \theta).
\]

**Proof.** Since \( f(z) \) and \( g(z) \) belongs to \( SRA(\lambda, \beta, \alpha, \mu, \theta) \), we have

\[
\sum_{n=2}^{\infty} \left\{ \frac{\beta + [n(1 + n\lambda - \lambda)(1 + \beta) - (\alpha + 2\beta)]} {1 - \alpha} K(n, \mu, \theta) b_n \right\} a_n \leq 1
\]
and
\[ \sum_{n=2}^{\infty} \left\{ \frac{\beta + [n(1 + n\lambda - \lambda)(n(1 + \beta) - (\alpha + 2\beta))]}{1 - \alpha} K(n, \mu, \theta) a_n \right\} b_n \leq 1 \]

and by applying the Cauchy-Schwarz inequality, we have
\[ \sum_{n=2}^{\infty} \left\{ \frac{\beta + [n(1 + n\lambda - \lambda)(n(1 + \beta) - (\alpha + 2\beta))]}{1 - \alpha} K(n, \mu, \theta) \sqrt{a_n b_n} \right\} \sqrt{a_n b_n} \leq \]
\[ \left( \sum_{n=2}^{\infty} \left\{ \frac{(1 - \lambda + n\lambda)(n(1 + \beta) - (\beta + \alpha))}{1 - \alpha} K(n, \mu, \theta) a_n \right\} \right)^{1/2} \]
\[ \left( \sum_{n=2}^{\infty} \left\{ \frac{(1 - \lambda + n\lambda)(n(1 + \beta) - (\beta + \alpha))}{1 - \alpha} K(n, \mu, \theta) b_n \right\} \right)^{1/2} \].

However, we obtain
\[ \sum_{n=2}^{\infty} \left\{ \frac{(1 - \lambda + n\lambda)(n(1 + \beta) - (\beta + \alpha))}{1 - \alpha} K(n, \mu, \theta) \sqrt{a_n b_n} \right\} \sqrt{a_n b_n} \leq 1 \]

Now, we have to prove that
\[ \sum_{n=2}^{\infty} \left\{ \frac{(1 - \lambda + n\lambda)(n(1 + \beta) - (\beta + \alpha))}{1 - \alpha} K(n, \mu, \theta) \right\} a_n b_n \leq 1 \]

Since
\[ \sum_{n=2}^{\infty} \left\{ \frac{(1 - \lambda + n\lambda)(n(1 + \beta) - (\beta + \alpha))}{1 - \alpha} K(n, \mu, \theta) a_n b_n \right\} \]
\[ = \sum_{n=2}^{\infty} \left\{ \frac{(1 - \lambda + n\lambda)(n(1 + \beta) - (\beta + \alpha))}{1 - \alpha} K(n, \mu, \theta) \sqrt{a_n b_n} \right\} \sqrt{a_n b_n} \].

Hence the proof.

4 Application of the Fractional Calculus

Various operators of fractional calculus (i.e. fractional derivative and fractional integral) have been rather extensively studied by many researchers ([5-7]) Each of these theorems would involve certain operator of fractional calculus which are defined as follows ([2]).
Definition 4.1. The fractional integral operator of order $\delta$ is defined, for a function $f(z)$, by

$$D^\delta_z(f(z)) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(t)}{(z-t)^{1-\delta}} \, dt, \quad \delta > 0$$ (4.1)

where $f(z)$ is analytic function in a simply connected region of $z$-plane containing the origin and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-t)$ to be read when $(z-t) > 0$.

Definition 4.2. The fractional derivative of order $\delta$ is defined for a function $f(z)$ by

$$D^\delta_z(f(z)) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\delta} \, dt, \quad 0 \leq \delta < 1.$$ (4.2)

where $f(z)$ is analytic function in a simply connected region of $z$-plane containing the origin and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-t)$ to be read when $(z-t) > 0$.

Definition 4.3. The fractional derivative of order $k+\delta$ is defined by

$$D^{k+\delta}_z(f(z)) = \frac{d^k}{dz^k} D^\delta_z f(z), \quad 0 \leq \delta < 1.$$ (4.3)

From definition 4.1 and 4.2, after a simple computation we obtain

$$D^{-\delta}_z f(z) = \frac{1}{\Gamma(2+\delta)} z^{\delta+1} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\delta)} a_n z^{n+\delta}$$ (4.4)

$$D^\delta_z f(z) = \frac{1}{\Gamma(2-\delta)} z^{1-\delta} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} a_n z^{n-\delta}$$ (4.5)

Now using equations (4.4) and (4.5), let us prove the following theorems:

Theorem 4.1. Let $f \in SRA(\lambda, \beta, \alpha, \mu, \theta)$. Then

$$|D^{-\delta}_z f(z)| \leq \frac{1}{\Gamma(2+\delta)} |z|^{\delta+1} \left[ 1 + \frac{2(1-\alpha)}{(2+\delta)[\beta+2(2-\alpha)(1+\lambda)(\theta+1)b_2]} |z|^{n-1} \right]$$ (4.6)

$$|D^\delta_z f(z)| \geq \frac{1}{\Gamma(2-\delta)} |z|^{1-\delta} \left[ 1 - \frac{2(1-\alpha)}{(2-\delta)[\beta+2(2-\alpha)(1+\lambda)(\theta+1)b_2]} |z|^{n-1} \right]$$ (4.7)

The inequalities (4.6) and (4.7) are attained for the function $f$ given by

$$f(z) = z - \frac{1-\alpha}{\beta+2(2-\alpha)(1+\lambda)(\theta+1)b_2} z$$ (4.8)


Proof. From Theorem (2.1), we obtain

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\alpha}{\beta+2(2-\alpha)(1+\lambda)(\theta+1)b_2}$$ (4.9)
Using equation (4.5), we obtain
\[ \Gamma(2 - \delta)z^\delta D_z^\delta f(z) = z - \sum_{n=2}^\infty \psi(n, \delta)a_n z^n \] (4.10)
such that
\[ \psi(n, \delta) = \frac{\Gamma(n + 1)\Gamma(2 + \delta)}{\Gamma(n + 1 + \delta)}, \quad n \geq 2. \]
where \( \psi(n, \delta) \) is a decreasing function of \( n \) and \( 0 < \psi(n, \delta) \leq (2, \delta) = \frac{2}{2 + \delta} \). Using equations (4.9) and (4.10), we obtain
\[ |\Gamma(2 - \delta)z^\delta D_z^\delta f(z)| \leq |z| + \psi(2, \delta)|z| \sum_{n=2}^\infty a_n \leq |z| + \frac{2(1 - \alpha)}{(2 + \delta)(\beta + 2(2 - \alpha)(1 + \lambda)(\theta + 1)b_2)z}, \]
which is an equation (4.6).

Similarly we can get equation (4.7).

**Theorem 4.2.** Let \( f \in SRA(\lambda, \beta, \alpha, \mu, \theta) \). Then
\[ |D_z^\delta f(z)| \leq \frac{1}{\Gamma(2 - \delta)}|z|^{1-\delta} \left[ 1 + \frac{2(1 - \alpha)}{(2 - \delta)[\beta + 2(2 - \alpha)(1 + \lambda)(\theta + 1)b_2]|z|^{n-1}} \right] \] (4.11)
\[ |D_z^\delta f(z)| \geq \frac{1}{\Gamma(2 - \delta)}|z|^{1-\delta} \left[ 1 - \frac{2(1 - \alpha)}{(2 - \delta)[\beta + 2(2 - \alpha)(1 + \lambda)(\theta + 1)b_2]|z|^{n-1}} \right] \] (4.12)
The inequalities (4.11) and (4.12) are attained for the function \( f(z) \) given by
\[ f(z) = z - \frac{1 - \alpha}{\beta + 2(2 - \alpha)(1 + \lambda)(\theta + 1)b_2}z \] (4.13)

**Proof.** From equation (4.5), we obtain
\[ \Gamma(2 - \delta)z^\delta D_z^\delta f(z) = z - \sum_{n=2}^\infty l(n, \delta)a_n z^n \]
such that
\[ l(n, \delta) = \frac{\Gamma(n + 1)\Gamma(2 - \delta)}{\Gamma(n + 1 + \delta)}, \quad n \geq 2. \] (4.14)
where \( l(n, \delta) \) is a decreasing function and \( 0 < l(n, \delta) \leq l(2, \delta) = \frac{2}{2 - \delta} \). Using equations (4.9) and (4.14), we obtain
\[ |\Gamma(2 - \delta)z^\delta D_z^\delta f(z)| \leq |z| + l(2, \delta)|z| \sum_{n=2}^\infty a_n \leq |z| + \frac{2(1 - \alpha)}{(2 + \delta)(\beta + 2(2 - \alpha)(1 + \lambda)(\theta + 1)b_2)|z|^{n-1}}. \]
which is nothing but equation (4.11). Similarly we can get equation (4.12)
Corollary 4.1. For every \( f \in SRA(\lambda, \beta, \alpha, \mu, \theta) \), we have
\[
\frac{|z|^2}{2} \left[ 1 - \frac{2(1-\alpha)}{3(\beta + 2(2-\alpha)(1+\lambda)(\theta + 1)b_2)} |z|^{n-1} \right] \leq \frac{1}{2} \left| \int_0^z f(t) \, dt \right|
\]
and
\[
|z| \left[ 1 - \frac{1-\alpha}{(\beta + 2(2-\alpha)(1+\lambda)(\theta + 1)b_2)} |z|^{n-1} \right] \leq |f(z)|
\]
\[
\leq |z| \left[ 1 + \frac{1-\alpha}{(\beta + 2(2-\alpha)(1+\lambda)(\theta + 1)b_2)} |z|^{n-1} \right]
\]

Proof. By Definition 4.1 and Theorem 4.1 for \( \delta = 1 \), we have \( D_z^{-1} f(z) = \int_0^z f(t) \, dt \), the result is true. Also by Definition 4.2 and Theorem 4.2 for \( \delta = 0 \), we have
\[
D^0 f(z) = \frac{d}{dz} \int_0^z f(t) \, dt = f(z)
\]
Hence the result is true.

H.M. Srivatsa et al. [5] have studied the certain sub class \( u(\lambda, \alpha, \beta, k) \) of normalized analytic functions in the open unit disk \( U \) which generalizes the familiar class of uniformly convex function.

5 Radii of class to convexity Theorem

Theorem 5.1. Let the function \( f(z) \) defined by \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \) be in the class \( SRA(\lambda, \beta, \alpha, \mu, \theta) \). Then \( f(z) \) is close to convex of order \( \alpha \) (0 < \( \alpha < 1 \)) in \( |z| < r_1(\lambda, \beta, \alpha, \mu, \theta, \rho) \) where
\[
r_1(\lambda, \beta, \alpha, \mu, \theta, \rho) = \inf \left\{ \frac{[\beta + z - \sum_{n=2}^{\infty} [n(1+n\lambda - \lambda)(n+1+\beta) - (\alpha + 2\beta)]K(n, \mu, \theta)b_n](1-\rho)}{n(1-\alpha)} \right\} \frac{1}{n-1}
\]

The result is sharp for the function \( f(z) \) given by
\[
f(z) = z - \frac{1-\alpha}{\beta + z - \sum_{n=2}^{\infty} [n(1+n\lambda - \lambda)(n+1+\beta) - (\alpha + 2\beta)]K(n, \mu, \theta)b_n} [z]^{n-1}
\]

Proof. It is sufficient to show that \( |f'(z) - 1| \leq 1 - \alpha \), 0 \( \leq \rho \leq 1 \) for \( |z| < r_1(\lambda, \beta, \alpha, \mu, \theta, \rho) \). We have
\[
|f'(z) - 1| = \left| - \sum_{n=2}^{\infty} na_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} na_n |z|^{n-1}
\]
Thus \( |f'(z) - 1| \leq 1 - \rho \) if \( \sum_{n=2}^{\infty} (\frac{n}{1-\rho})a_n |z|^{n-1} \leq 1 \).
Hence by theorem(1), \( \frac{z^n}{1-\beta} \) will be true if

\[
\left( \frac{n}{1-\rho} \right) |z|^{n-1} \leq \frac{\beta + z - \sum_{n=2}^{\infty} [n(1+n\lambda - \lambda)(n(1+\beta) - (\alpha+2\beta))] K(n, \mu, \theta)b_n}{1-\alpha}
\]

(or) if

\[
|z| \leq \left\{ \frac{\beta + z - \sum_{n=2}^{\infty} n(1+n\lambda - \lambda)(n(1+\beta) - (\alpha+2\beta)) K(n, \mu, \theta)b_n(1-\rho)}{n(1-\alpha)} \right\} \frac{1}{n-1}
\]

The theorem follows easily from previous equation.

References


Received: September, 2012