On the Inverse Diamond Kernel of Marcel Riesz

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Abstract

In this paper, we define the diamond Marcel Riesz operator of order \((\alpha, \beta)\) on the function \(f\) by

\[ M^{(\alpha, \beta)}(f) = K^{\alpha, \beta} * f, \]

where \(K^{\alpha, \beta}\) is diamond kernel of Marcel Riesz, \(\alpha, \beta \in \mathbb{C}\), the symbol \(*\) designates the convolution, and \(f \in \mathcal{S}\), \(\mathcal{S}\) is the Schwartz space of functions. Our purpose of this paper is to obtain the operator \(N^{(\alpha, \beta)} = [M^{(\alpha, \beta)}]^{-1}\) such that if \(M^{(\alpha, \beta)}(f) = \varphi\), then \(N^{(\alpha, \beta)} \varphi = f\). Our results generalize formulae appearing in A. Kananthai [On the convolutions of the diamond kernel of Marcel Riesz, Applied Mathematics and Computation, 114(2000), 95 – 101].

Mathematics Subject Classification: 46F10, 46F12

Keywords: Diamond kernel of Marcel Riesz, diamond operator, ultra-hyperbolic kernel of Marcel Riesz, ultra-hyperbolic operator, Dirac-delta distribution

1 Introduction

The \(n\)-dimensional ultra-hyperbolic operator \(\Box^k\) iterated \(k\) times is defined by

\[ \Box^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \]

where \(p + q = n\) is the dimension of \(\mathbb{R}^n\) and \(k\) is a non-negative integer.

Consider the linear differential equation in the form of

\[ \Box^k u(x) = f(x), \]

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where \( u(x) \) and \( f(x) \) are generalized functions and \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \).

I. M. Gelfand and G. E. Shilov [4] were the first to introduce the fundamental solution of (2), which is a complicated form. Later, S. E. Trione [19] showed that the generalized function \( R_{2k}(x) \), defined by (14) with \( \gamma = 2k \), is the unique fundamental solution of (2) and M. A. Tellez [15] also proved that \( R_{2k}(x) \) exists only when \( p + q = n \) with odd \( p \).

Later, A. Kananthai [10] was the first to introduce the operator \( \diamond^k \) called the diamond operator iterated \( k \) times, which is defined by

\[
\diamond^k = \left[ \left( \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left( \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k,
\]

where \( n = p + q \) is the dimension of \( \mathbb{R}^n \), for all \( x = (x_1, x_2, \ldots, x_n) \), and \( k \) is a non-negative integer. The operator \( \diamond^k \) can be expressed in the form

\[
\diamond^k = \triangle^k \square^k = \square^k \triangle^k,
\]

(4)

where \( \square^k \) is defined by (1), and

\[
\triangle^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k.
\]

is the Laplace operator iterated \( k \) times. On finding the fundamental solution of this product, A. Kananthai used the convolution of functions which are fundamental solutions of the operators \( \square^k \) and \( \triangle^k \). He found that the convolution

\[
(-1)^k S_{2k}(x) * R_{2k}(x)
\]

is the fundamental solution of the operator \( \diamond^k \), that is,

\[
\diamond^k ( (-1)^k S_{2k}(x) * R_{2k}(x) ) = \delta(x),
\]

(6)

where \( R_{2k}(x) \) and \( S_{2k}(x) \) are defined by (14) and (19), respectively, with \( \gamma = 2k \), and \( \delta(x) \) is the Dirac delta distribution. The fundamental solution

\[
(-1)^k S_{2k}(x) * R_{2k}(x)
\]

is called the diamond kernel of Marcel Riesz. A wealth of some effective works on the diamond kernel of Marcel Riesz were presented by A. Kananthai [5, 6, 7, 8, 9, 14, 18].

In 1978, A. G. Dominguez and S. E. Trione [3] introduced the distributional functions \( H_\alpha(P \pm i0, n) \) which are causal (anti-causal) analogues of the elliptic kernel of Riesz [12]. Later, R. A. Cerutti and S. E. Trione [2] defined the causal (anti-causal) generalized Marcel Riesz potentials of order \( \alpha, \alpha \in \mathbb{C} \), by

\[
R^\alpha \varphi = H_\alpha(P \pm i0, n) * \varphi,
\]

(7)

where \( \varphi \in \mathcal{S}, \mathcal{S} \) is the Schwartz space of functions [13], and \( H_\alpha(P \pm i0, n) \) is given by

\[
H_\alpha(P \pm i0, n) = \frac{e^{\mp \alpha \pi i/2} e^{\pm \pi i/2} \Gamma((n - \alpha)/2)(P \pm i0)^{(\alpha - n)/2}}{2^{\alpha} \pi^{n/2} \Gamma(\alpha/2)}.
\]

(8)
Here, $P$ is defined by
\begin{equation}
P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2,
\end{equation}
where $q$ is the number of negative terms of the quadratic form $P$. The distributions $(P \pm i\lambda)$ are defined by
\begin{equation}
(P \pm i\lambda) = \lim_{\epsilon \to 0} (P \pm i\epsilon |x|^2)^\lambda,
\end{equation}
where $\epsilon > 0$, $\lambda \in \mathbb{C}$ and $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, see [4]. They also studied the inverse operator of $R^\alpha$, denoted by $(R^\alpha)^{-1}$, such that if $f = R^\alpha \varphi$, then $(R^\alpha)^{-1} f = \varphi$.

Later, M. A. Aguirre [1] defined the ultra-hyperbolic Marcel Riesz operator $M^\alpha$ of the function $f$ by
\begin{equation}
M^\alpha(f) = R^\alpha * f,
\end{equation}
where $R^\alpha$ is defined by (14) and $f \in \mathcal{S}$. He also studied the operator $N^\alpha = (M^\alpha)^{-1}$ such that if $M^\alpha(f) = \varphi$, then $N^\alpha \varphi = f$.

Let us consider the diamond kernel of Marcel Riesz $K_{\alpha,\beta}(x)$ introduced by A. Kananthai [6], which is given by the convolution
\begin{equation}
K_{\alpha,\beta}(x) = S_\alpha * R_\beta,
\end{equation}
where $S_\alpha$ is elliptic kernel defined by (19) and $R_\beta$ is the ultra-hyperbolic kernel defined by (14). M. A. Tellez and A. Kananthai [18] proved that $K_{\alpha,\beta}(x)$ exists and is in the space of rapidly decreasing distributions. Moreover, they also showed that the convolution of the distributional families $K_{\alpha,\beta}(x)$ relates to the diamond operator.

In this paper, we define the diamond Marcel Riesz operator of order $(\alpha, \beta)$ of the function $f$ by
\begin{equation}
M^{(\alpha,\beta)}(f) = K_{\alpha,\beta} * f,
\end{equation}
where $K_{\alpha,\beta}$ is defined by (12), $\alpha, \beta \in \mathbb{C}$, the symbol $\ast$ designates the convolution, and $f \in \mathcal{S}$. Our aim of this paper is to obtain the operator $N^{(\alpha,\beta)} = [M^{(\alpha,\beta)}]^{-1}$ such that if $M^{(\alpha,\beta)}(f) = \varphi$, then $N^{(\alpha,\beta)} \varphi = f$.

Before we proceed to our main theorem, the following definitions and some concepts require some clarifications.

### 2 Preliminary Notes

**Definition 2.1.** Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the $n$-dimensional Euclidean space $\mathbb{R}^n$. Let
\begin{equation}
u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2,
\end{equation}
be the nondegenerated quadratic form, where \( p + q = n \) is the dimension of \( \mathbb{R}^n \). Let \( \Gamma_+ = \{ x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0 \} \) be the interior of a forward cone and let \( \overline{\Gamma}_+ \) denote its closure. For any complex number \( \gamma \), we define

\[
R_\gamma(x) = \begin{cases} \frac{u^{(\gamma-n)/2}}{K_n(\gamma)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases}
\]

(14)

where

\[
K_n(\gamma) = \frac{\pi^{(n-1)/2} \Gamma((2 + \gamma - n)/2) \Gamma((1 - \gamma)/2) \Gamma(\gamma)}{\Gamma((2 + \gamma - p)/2) \Gamma((p - \gamma)/2)}.
\]

(15)

The function \( R_\gamma(x) \) is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Nozaki [11]. It is well known that \( R_\gamma(x) \) is an ordinary function if \( \Re(\gamma) \geq n \) and is a distribution of \( \gamma \) if \( \Re(\gamma) < n \). Let \( \text{supp } R_\gamma(x) \) denote the support of \( R_\gamma(x) \) and suppose that \( \text{supp } R_\gamma(x) \subset \overline{\Gamma}_+ \) (i.e. \( \text{supp } R_\gamma(x) \) is compact).

By putting \( p = 1 \) in \( R_{2k}(x) \) and taking into account Legendre’s duplication formula for \( \Gamma(z) \), that is,

\[
\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma(z + 1/2),
\]

(16)

we obtain

\[
I_H^\gamma(x) = \frac{v^{(\gamma-n)/2}}{H_n(\gamma)},
\]

(17)

and \( v = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2 \), where

\[
H_n(\gamma) = \pi^{(n-2)/2} 2^{\gamma-1} \Gamma((\gamma + 2 - n)/2) \Gamma(\gamma/2).
\]

(18)

The function \( I_H^\gamma(x) \) is called the hyperbolic kernel of Marcel Riesz.

**Definition 2.2.** Let \( x = (x_1, x_2, \ldots, x_n) \) be a point of \( \mathbb{R}^n \) and \( \omega = x_1^2 + x_2^2 + \cdots + x_n^2 \). The elliptic kernel of Marcel Riesz is defined by

\[
S_\gamma(x) = \frac{\omega^{(\gamma-n)/2}}{W_n(\gamma)},
\]

(19)

where \( n \) is the dimension of \( \mathbb{R}^n \), \( \gamma \in \mathbb{C} \), and

\[
W_n(\gamma) = \frac{\pi^{n/2} 2^{\gamma} \Gamma(\gamma/2)}{\Gamma((n - \gamma)/2)}.
\]

(20)

Note that \( n = p + q \). By putting \( q = 0 \) (i.e. \( n = p \)) in (14) and (15), we can reduce \( u^{(\gamma-n)/2} \) to \( \omega_p^{(\gamma-p)/2} \), where \( \omega_p = x_1^2 + x_2^2 + \cdots + x_p^2 \), and reduce \( K_n(\gamma) \) to

\[
K_p(\gamma) = \frac{\pi^{(p-1)/2} \Gamma((1 - \gamma)/2) \Gamma(\gamma)}{\Gamma((p - \gamma)/2)}.
\]
Using Legendre’s duplication formula
\[ \Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma(z + 1/2), \]  
and
\[ \Gamma(1/2 + z) \Gamma(1/2 - z) = \pi \sec(\pi z), \]
we obtain
\[ K_p(\gamma) = \frac{1}{2} \sec\left(\frac{\gamma \pi}{2}\right) W_p(\gamma). \]

Thus, for \( q = 0 \), we have
\[ R_0(\gamma)(x) = \frac{u(\gamma - p)}{K_p(\gamma)} = 2 \cos\left(\frac{\gamma \pi}{2}\right) W_p(\gamma) = 2 \cos\left(\frac{\gamma \pi}{2}\right) S_\gamma(x). \]

In addition, if \( \gamma = 2k \) for some non-negative integer \( k \), then
\[ R_{2k}(x) = 2(-1)^k S_{2k}(x). \]

The proofs of Lemmas 2.3 and 2.4 are given in \[18\].

**Lemma 2.3.** The function \( K_{\alpha,\beta}(x) \) has the following properties:

(i) \( K_{0,0}(x) = \delta(x) \);

(ii) \( K_{-2k,-2k}(x) = (-1)^k \delta(x) \);

(iii) \( \delta^k(K_{\alpha,\beta}(x)) = (-1)^k K_{-2k,\beta-2k}(x) \);

(iv) \( \delta^k(K_{2k,2k}(x)) = (-1)^k \delta(x) \);

(v) \( K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = (-1)^k \delta^k K_{\alpha,\beta}(x) \).

**Lemma 2.4.** (The convolutions of \( K_{\alpha,\beta}(x) \))

(i) If \( p \) is odd, then
\[ K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = (R_{\beta+\beta'} + T_{\beta,\beta'}) * S_{\alpha+\alpha'}, \]
where \( R_\beta \) and \( S_\alpha \) are defined by (14) and (19), respectively. \( T_{\beta,\beta'} \) is defined by
\[ T_{\beta,\beta'} = -\frac{i}{2} \frac{\sin(\beta \pi/2) \sin(\beta' \pi/2)}{\sin((\beta + \beta') \pi/2)} \left[ H^+_{\beta+\beta'} - H^-_{\beta+\beta'} \right] \]
and
\[ H^\pm_q = H_q(P \pm i0, n) \]
as defined by (8).
(ii) If \( p \) is even, then
\[
K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = B_{\beta,\beta'} R_{\beta+\beta'} * S_{\alpha+\alpha'},
\]
(29)
where
\[
B_{\beta,\beta'} = \frac{\cos(\beta \pi/2) \cos(\beta' \pi/2)}{\cos ((\beta + \beta') \pi/2)}.
\]

3 The Convolution \( K_{\alpha,\beta} * K_{\alpha',\beta'} \) when \( \alpha' = -\alpha, \beta' = -\beta \)

Now, we consider the property of \( K_{\alpha,\beta} * K_{\alpha',\beta'} \) when \( \alpha' = -\alpha \) and \( \beta' = -\beta \).

From (26) and (29), we know that the following properties are valid:

1. If \( p \) is odd and \( q \) is even, then
\[
K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = (R_{\beta+\beta'} + T_{\beta,\beta'}) * S_{\alpha+\alpha'},
\]
(30)
where \( R_{\beta}, S_{\alpha} \) and \( T_{\beta,\beta'} \) are defined by (14), (19), and (27), respectively.

2. If \( p \) and \( q \) are both odd, then
\[
K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = (R_{\beta+\beta'} + T_{\beta,\beta'}) * S_{\alpha+\alpha'}.
\]
(31)

3. If \( p \) is even and \( q \) is odd, then
\[
K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = \cos(\beta \pi/2) \cos(\beta' \pi/2) \cos((\beta + \beta') \pi/2) R_{\beta+\beta'} * S_{\alpha+\alpha'}.
\]
(32)

4. If \( p \) and \( q \) are both even, then
\[
K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = \frac{\cos(\beta \pi/2) \cos(\beta' \pi/2)}{\cos((\beta + \beta') \pi/2)} R_{\beta+\beta'} * S_{\alpha+\alpha'}.
\]
(33)

Moreover, it follows from (27) that
\[
T_{\beta,\beta'} = \lim_{\beta' \to -\beta} T_{\beta,\beta'} = -\frac{i}{2} \lim_{\gamma \to 0} \frac{\sin(\beta \pi/2) \sin((\gamma - \beta) \pi/2)}{\sin(\gamma \pi/2)} [H_{\gamma}^+ - H_{\gamma}^-]
\]
\[
= -\frac{i}{2} \lim_{\gamma \to 0} \frac{\sin(\beta \pi/2) \sin((\gamma - \beta) \pi/2)}{\sin(\gamma \pi/2)} \cdot \lim_{\gamma \to 0} [H_{\gamma}^+ - H_{\gamma}^-],
\]
(34)
where \( \gamma = \beta + \beta' \).
On the other hand, using (28) and (8), we have

\[
\lim_{\gamma \to 0} [H_{\gamma}^+ - H_{\gamma}^-] = \frac{\Gamma(n/2)}{\pi^{n/2}} \left[ \lim_{\gamma \to 0} \frac{e^{-\gamma \pi i/2} e^{q \pi i/2}}{\Gamma(\gamma/2)} \frac{(P + i0)^{(\gamma-n)/2}}{\Gamma(\gamma/2)} - \lim_{\gamma \to 0} \frac{e^{\gamma \pi i/2} e^{-q \pi i/2}}{\Gamma(\gamma/2)} \frac{(P - i0)^{(\gamma-n)/2}}{\Gamma(\gamma/2)} \right] = \frac{\Gamma(n/2)}{\pi^{n/2}} \left[ \lim_{\gamma \to 0} \frac{e^{-\gamma \pi i/2} e^{q \pi i/2}}{\Gamma(\gamma/2)} \frac{\text{Res}_{\beta=-(n/2)} (P + i0)^{\beta}}{\text{Res}_{\beta=-(n/2)} \Gamma(\beta + n/2)} - \lim_{\gamma \to 0} \frac{e^{\gamma \pi i/2} e^{-q \pi i/2}}{\Gamma(\gamma/2)} \frac{\text{Res}_{\beta=-(n/2)} (P - i0)^{\beta}}{\text{Res}_{\beta=-(n/2)} \Gamma(\beta + n/2)} \right].
\]

(35)

Now, taking \( n \) as an odd integer, we obtain

\[
\text{Res}_{\lambda=-n/2-k} (P \pm i0)^{\lambda} = \frac{e^{\pm q \pi i/2} \pi^{n/2}}{2^{2k} k! \Gamma(n/2 + k)} \square^k \delta(x),
\]

where \( \square^k \) is defined by (1), \( p + q = n \), and \( k \) is non-negative integer; see [16, 17]. If \( p \) and \( q \) are both even, then

\[
\text{Res}_{\lambda=-n/2-k} (P \pm i0)^{\lambda} = \frac{e^{\pm q \pi i/2} \pi^{n/2}}{2^{2k} k! \Gamma(n/2 + k)} \square^k \delta(x).
\]

(36)

Nevertheless, if \( p \) and \( q \) are both odd, then

\[
\text{Res}_{\lambda=-n/2-k} (P \pm i0)^{\lambda} = 0,
\]

(38)

Therefore, we have

\[
\lim_{\gamma \to 0} [H_{\gamma}^+ - H_{\gamma}^-] = \frac{\Gamma(n/2)}{\pi^{n/2}} \frac{\pi^{n/2}}{\Gamma(n/2)} \left[ \lim_{\gamma \to 0} e^{-\gamma \pi i/2} - \lim_{\gamma \to 0} e^{\gamma \pi i/2} \right] \delta(x) = \lim_{\gamma \to 0} \left[ -2i \sin(\gamma \pi/2) \right] \delta(x).
\]

(39)

From (35) and (38), we have

\[
\lim_{\gamma \to 0} [H_{\gamma}^+ - H_{\gamma}^-] = 0
\]

if \( p \) and \( q \) are both odd (\( n \) even).

Applying (39) and (40) in (34), we have

\[
T_{\beta,-\beta} = -\frac{i}{2} \lim_{\gamma \to 0} \frac{\sin(\beta \pi/2) \sin((\gamma - \beta) \pi/2)}{\sin(\gamma \pi/2)} \cdot \lim_{\gamma \to 0} \left[ -2i \sin(\gamma \pi/2) \right] \delta(x) = \sin^2(\beta \pi/2) \delta(x)
\]

(41)
if $p$ is odd and $q$ is even, and

$$T_{\beta,-\beta} = 0 \tag{42}$$

if $p$ and $q$ are both odd.

From (30) to (33) and using Lemmas 2.3 and 2.4 and the formulae (41) and (42), if $p$ is odd and $q$ is even, then we obtain

$$K_{\alpha,\beta} \ast K_{-\alpha,-\beta} = (R_0 + T_{\beta,-\beta}) \ast S_0$$

$$= \left[ \delta(x) + \sin^2(\beta \pi/2) \delta(x) \right] \ast \delta(x)$$

$$= \delta(x) + \sin^2(\beta \pi/2) \delta(x)$$

$$= [1 + \sin^2(\beta \pi/2)] \delta(x). \tag{43}$$

If $p$ and $q$ are both odd, then

$$K_{\alpha,\beta} \ast K_{-\alpha,-\beta} = (R_0 + T_{\beta,-\beta}) \ast S_0$$

$$= R_0 \ast S_0$$

$$= K_{0,0} = \delta(x). \tag{44}$$

If $p$ is even and $q$ is odd, then

$$K_{\alpha,\beta} \ast K_{-\alpha,-\beta} = \frac{\cos(\beta \pi/2) \cos(-\beta \pi/2)}{\cos((\beta - \beta) \pi/2)} (R_0 \ast S_0)$$

$$= \cos^2(\beta \pi/2) \delta(x). \tag{45}$$

Finally, if $p$ and $q$ are both even, then

$$K_{\alpha,\beta} \ast K_{-\alpha,-\beta} = \frac{\cos(\beta \pi/2) \cos(-\beta \pi/2)}{\cos((\beta - \beta) \pi/2)} (R_0 \ast S_0)$$

$$= \cos^2(\beta \pi/2) \delta(x). \tag{46}$$

4 The Main Theorem

Let $M^{(\alpha,\beta)}(f)$ be the diamond Marcel Riesz operator of order $(\alpha, \beta)$ of the function $f$, which is defined by

$$M^{(\alpha,\beta)}(f) = K_{\alpha,\beta} \ast f, \tag{47}$$

where $K_{\alpha,\beta}$ is defined by (12), $\alpha, \beta \in \mathbb{C}$, and $f \in \mathcal{S}$.

Recall that our objective is to obtain the operator $N^{\alpha,\beta} = [M^{(\alpha,\beta)}]^{-1}$ such that if $M^{(\alpha,\beta)}(f) = \varphi$, then $N^{(\alpha,\beta)} \varphi = f$ for all $\alpha, \beta \in \mathbb{C}$.

We are now ready to state our main theorem.
Theorem 4.1. If $M^{(\alpha,\beta)}(f) = \varphi$ (where $M^{(\alpha,\beta)}(f)$ is defined by (47) and $f \in \mathcal{S}$), then $N^{(\alpha,\beta)} \varphi = f$ such that

$$N^{(\alpha,\beta)} = [M^{(\alpha,\beta)}]^{-1} = \begin{cases} [1 + \sin^2(\beta \pi/2)]^{-1} K_{-\alpha,-\beta} & \text{if } p \text{ is odd and } q \text{ is even;} \\ K_{-\alpha,-\beta} & \text{if } p \text{ and } q \text{ are both odd;} \\ \sec^2(\beta \pi/2) K_{-\alpha,-\beta} & \text{if } p \text{ is even with } \beta \neq 2s + 1 \end{cases}$$

for any non-negative integer $s$.

Proof. By (47), we have

$$M^{(\alpha,\beta)}(f) = K_{\alpha,\beta} \ast f = \varphi,$$

where $K_{\alpha,\beta}$ is defined by (12), $\alpha, \beta \in \mathbb{C}$ and $f \in \mathcal{S}$. If $p$ is odd and $q$ is even, then, in view of (43), we obtain

$$[1 + \sin^2(\beta \pi/2)]^{-1} K_{-\alpha,-\beta} \ast (K_{\alpha,\beta} \ast f) = [1 + \sin^2(\beta \pi/2)]^{-1} (K_{-\alpha,-\beta} \ast K_{\alpha,\beta}) \ast f = [1 + \sin^2(\beta \pi/2)]^{-1} \{ [1 + \sin^2(\beta \pi/2)] \delta(x) \} \ast f = \delta \ast f = f.$$

Hence,

$$[1 + \sin^2(\beta \pi/2)]^{-1} K_{-\alpha,-\beta} = [M^{(\alpha,\beta)}]^{-1} = (K_{\alpha,\beta})^{-1} \tag{48}$$

for all $\alpha, \beta \in \mathbb{C}$.

Similarly, if both $p$ and $q$ are odd, then by (44), we obtain

$$K_{-\alpha,-\beta} \ast (K_{\alpha,\beta} \ast f) = (K_{-\alpha,-\beta} \ast K_{\alpha,\beta}) \ast f = \delta \ast f = f.$$

Hence,

$$K_{-\alpha,-\beta} = [M^{(\alpha,\beta)}]^{-1} = (K_{\alpha,\beta})^{-1} \tag{49}$$

for all $\alpha, \beta \in \mathbb{C}$.

Finally, if $p$ is even, then by (45) and (46), we have

$$\sec^2(\beta \pi/2) K_{-\alpha,-\beta} \ast (K_{\alpha,\beta} \ast f) = \sec^2(\beta \pi/2) (K_{-\alpha,-\beta} \ast K_{\alpha,\beta}) \ast f = \sec^2(\beta \pi/2) \{ \cos^2(\beta \pi/2) \delta(x) \} \ast f = \delta \ast f = f,$$

provided that $\beta \neq 2s + 1$ for any non-negative integer $s$.

Hence,

$$\sec^2(\beta \pi/2) K_{-\alpha,-\beta} = [M^{(\alpha,\beta)}]^{-1} = (K_{\alpha,\beta})^{-1} \tag{50}$$

for all $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 2s + 1$ for any non-negative integer $s$.

In the conclusion, formulae (48), (49) and (50) are the desired results, and this completes the proof.

Acknowledgement

The second author would like to thank Khon Kaen University for financial support.
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Received: October, 2012