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On Common Fixed Point in Probabilistic Metric Space

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Abstract

In this paper, we establish a new common fixed point theorem for four mappings in probabilistic metric spaces. The obtained result is a generalization of some fixed point theorems as in [1], [2], [6] and anthers.

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1 Introduction and Preliminaries

In 1942 K. Menger [3] introduce the notion of probabilistic metric space, was developed by numerous authors, especially Shweiser and Sklar ([4],[5]). The idea of Menger was to replace the distance $d(x, y)$ between two points x and y

with a distribution function F_{xy} that maps each positive real t, the probability that the distance $d(x, y)$ is less than or equal to t; and denoted by $F_{xy}(t)$. Such a probabilistic generalization of metric space appears to be well adapted for the investigation of psychology [6] and physical [9]. It is also of fundamental importance in probabilistic functional analysis.
In the sequel, the space of distance distribution functions(briefly d.d.f) is $\triangle^+ = \{f : [0, \infty] \to [0, 1] : f$ is left continuous on $(0, \infty)$, nondecreasing, $f(0) = 0$ and $f(\infty) = 1\}$. The subset $D^+ \subseteq \triangle^+$ is the set

$$D^+ = \{f \in \triangle^+ : L^- f(\infty) = 1\}.$$

Here $L^- f(t)$ denotes the left limit of the function f at the point t.
For $a \in [0, \infty)$, the element $\epsilon_a \in D^+$ is defined as

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if} \quad x \leq a \\ 1 & \text{if} \quad x > a \end{cases}$$

and

$$\varepsilon_\infty(x) = \begin{cases} 0, & 0 \leq x < \infty, \\ 1, & x = \infty. \end{cases}$$

By setting $f \leq g$ whenever $f(x) \leq g(x)$ for all $x \in [0, \infty)$, one introduces a natural ordering in $\triangle^+$, in this ordering the d.d.f ϵ_0 is the maximal of $\triangle^+$. Convergence in $\triangle^+$ is assumed to be weakly convergence, i.e $f_n \to f$ if and only if $f_n(x) \to f(x)$ at each continuity point x of f.
A t-norm is a binary operation on $[0, 1]$ which is associative, commutative, nondecreasing in each place and has 1 as identity. Two typical examples of continuous t-norms are:

$$T_p(a, b) = ab \quad , \quad T_M(a, b) = Min(a, b) \text{ and } \quad T_L(a, b) = max\{a + b - 1, 0\}.$$

A triangle function is a mapping $\tau : \triangle^+ \times \triangle^+ \to \triangle^+$ that is associative, commutative, nondecreasing in each place and has ϵ_0 as identity. Typical continuous triangle function is

$$\tau_T(f, g)(t) = sup\{T(f(u), g(v)) : u + v = t\}.$$

where T is a continuous t-norm.

Definition 1.1 *A probabilistic metric space (briefly,PM space) is a triple (X, F, τ) where X is a nonempty set , F is a function from $X \times X$ into $\triangle^+$, τ is a continuous triangle function , and the following conditions are satisfied for all x, y, z in X,*
(i) $F(x, x) = \varepsilon_0$.
(ii) $F(x, y) \neq \varepsilon_0$ *if* $x \neq y$.
(iii) $F(x, y) = F(y, x)$.
(iv) $F(x, z) \geq \tau(F(x, y), F(y, z))$.

Throughout this paper, we shall frequently denoted $F(x,y)$ by F_{xy}.

Definition 1.2 *Let (M,F) be a probabilistic semimetric space (i.e. (i), (ii) and (iii) are satisfied). For p in M and $t>0$, the strong t-neighborhood of p is the set*

$$N_p(t)=\{q\in M:\ F_{pq}(t)>1-t\}.$$

and the strong neighborhood system for M is

$$\{N_p(t);\ p\in M,\ t>0\}.$$

Lemma 1.3 [4] *Let (M,F,τ) be a PM space. If τ is continuous, then the family Υ consisting of $\emptyset$ and all unions of elements of strong neighborhood system for M determines a Hausdorff topology for M.*

An immediate consequence of Lemma 1.3 is that the family $\{N_p(t):t>0\}$ is a neighborhood system

Definition 1.4 [4] *Let $\{x_n\}$ be a sequence in a PM space (X,F,τ). Then*
(i) *The sequence $\{x_n\}$ is said to be convergent to $x\in X$ if for all $t>0$ there exist a positif integer N such that $F_{x_nx}(t)>1-t$ for $n\geq N$.*
(ii) *The sequence $\{x_n\}$ is called a Cauchy sequence if for all $t>0$ there exist a positif integer N such that $F_{x_nx_m}(t)>1-t$ for $n,m\geq N$.*
(iii) *A PM space (X,F,τ) is said to be complete if each Cauchy sequence in X is convergent to some point x in X.*

Lemma 1.5 [4] *Let $\{x_n\}$ be a sequence in a PM space (X,F,τ). Then*
(i) *The sequence $\{x_n\}$ to be convergent to $x\in X$ iff $\lim_{n\to\infty}F_{x_nx}=\varepsilon_0$.*
(ii) *The sequence $\{x_n\}$ is a Cauchy sequence iff $\lim_{n,m\to\infty}F_{x_nx_m}=\varepsilon_0$.*

Lemma 1.6 [4] *If (X,F,τ) is a PM space, (x_n) and (y_n) are sequences such that $x_n\to x$ and $y_n\to y$, then $F_{x_ny_n}\to F_{xy}$.*

Here and in the sequel, when we speak about a probabilistic metric space (M,F,τ), we always assume that τ is continuous and M be endowed with the topology Υ.

Recall the Definition of probabilistic diameter of a set in PM space.

Definition 1.7 [4] *Let A a nonempty subset of a PM space (X,F,τ). The probabilistic diameter of A is the function defined on $[0,\infty]$ by $D_A(\infty)=1$ and $D_A(t)=L^-\varphi_A(t)$ on $[0,\infty)$. Where*

$$\varphi_A(t)=\inf\{F_{pq}(t)|p,q\quad in\quad A\}$$

It is immediate that D_A is in $\triangle^+$ for any $A \subseteq X$, and for all p, q in A, $F_{pq} \geq D_A$.

Definition 1.8 [4] *A nonempty set A in a PM space is*
(i) *Bounded if* D_A *is in* D^+.
(ii) *Semi-bounded if* $0 < \lim_{t\to\infty} D_A(t) < 1$.
(iii) *Unbounded if* $\lim_{t\to\infty} D_A(t) = 0$.

Definition 1.9 *We denoted* Φ *the set of all function* ϕ *satisfies the booths conditions:*
(i) $\phi : [0,\infty] \to [0,\infty]$ *lower semi-continuous from the left, nondecreasing and* $\phi(0) = 0$.
(ii) $\phi(t) > t$ *for all* $t > 0$.

Lemma 1.10 [2] *Let* $\phi \in \Phi$. *Then*
(i) $\lim_{n\to\infty} \phi^n(s) = \infty$ *for all* $s > 0$.
(ii) *For all* $g \in D^+$ *and* $s > 0$ $\lim_{n\to\infty} g(\phi^n(s)) = 1$.

Definition 1.11 *Let* (X, F, τ) *be a PM space. A map S from X into itself, is said to be continuous, if every sequence* (p_n) *be convergent to p, the sequence* $S(p_n)$ *be converge to* $S(p)$.

Definition 1.12 *Let* (X, F, τ) *be a PM space. Let's* A, B, S *and* T *are mappings from X into itself. A sequence* (y_n) *is to be said compatible with* $[A, B, T, S]$ *in* $x_0 \in X$, *if there exist a sequence* $(x_n) \subset X$ *where*

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+2} = Sx_{2n+2} = Bx_{2n+1} \quad \text{for} \quad n = 0, 1, 2.....$$

Example 1.13 *Let* (X, F, τ) *be a PM space. Let's* A, B, S *and* T *are mappings from X into itself such that* $AX \subseteq TX$ *and* $BX \subseteq SX$.
Let $x_0 \in X$, *as* $Ax_0 \in TX$, *we can choose* $x_1 \in X$ *which* $Ax_0 = Tx_1$.
Since $Bx_1 \in SX$, *there exist* $x_2 \in X$ *where* $Sx_2 = Bx_1$.
Inductively we construct a sequence $(x_n) \subset X$ *satisfies*

$$Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad Sx_{2n+2} = Bx_{2n+1} \quad \text{for} \quad n = 0, 1, 2.....$$

Putting $y_{2n+1} = Tx_{2n+1}$ *and* $y_{2n+2} = Sx_{2n+2}$ *for* $n = 0.1.2........$
So (y_n) *is a sequence compatible with* $[A, B, T, S]$ *in* x_0.

Definition 1.14 *Let* (X, F, τ) *be a PM space and* $A, B : X \to X$.
(i) *The pair* $[A, B]$ *is called to be compatible, if for every sequence* (x_n) *in X which* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \in X$, *we have* $F_{ABx_nBAx_n} \longrightarrow \epsilon_0$.
(ii) *The pair* $[A, B]$ *to be said commute, if* $ABx = BAx$ *for all x.*

Remark 1.15 *If A and B are commute maps, then* $[A, B]$ *is compatible.*

Lemma 1.16 *Let (X, F, τ) be a PM space and $A, B : X \to X$ where B is continuous, and let (x_n) be a sequence in X such that $Ax_n \to z$ and $Bx_n \to z$ which $z \in X$. If the pair $[A, B]$ is compatible, then $ABx_n \to Bz$.*

Proof. Since

$$F_{ABx_nBz} \geq \tau(F_{ABx_nBAx_n}, F_{BAx_nBz})$$

Letting $n \to \infty$, using $Ax_n \to z$ and continuity of B and the fact that $F_{ABx_nBAx_n} \to \varepsilon_0$, we obtain

$$F_{ABx_nBz} \longrightarrow \varepsilon_0,$$

that is

$$ABx_n \to Bz.$$

Lemma 1.17 *Let (X, F, τ) be a PM space which $RanF \subseteq D^+$ and let x, y in X. If there exist $\phi \in \Phi$ such that $F_{xy}(t) \geq F_{xy}(\phi(t))$ for all $t > 0$, then $x = y$.*

Proof. Let $t > 0$, as $F_{xy}(t) \geq F_{xy}(\phi(t))$ and inductively, we have $F_{xy}(t) \geq F_{xy}(\phi^n(t))$ for $n = 1.2.3......$
Taking $n \to \infty$, utilizing Lemma 1.10, we obtain

$$F_{xy} = \epsilon_0.$$

So $x = y$.

Definition 1.18 *Let (X, F, τ) be a PM space. Let A, B, S and T are mappings from X into itself.*
(i) *The pair $[A, B]$ to be called (S, T)-Boyd-Wong contraction, if there exist $\phi \in \Phi$ such that:*

$$F_{AxBy}(t) \geq F_{SxTy}(\phi(t)),$$

for all x, y in X and $t > 0$.
(ii) *If $A = B$, then A is (S, T)-Boyd-Wong contraction.*

Definition 1.19 *Let (X, F, τ) be a PM space and $A, S, T : X \to X$ which $AX \subset TX \cap SX$ and let $x_0 \in X$.*
(i) *The (S, T)-orbit of A in x_0 is the set*

$$\{Ax_n : Ax_{2n} = Tx_{2n+1} \quad and \quad Ax_{2n+1} = Sx_{2n+2} \quad for \quad n = 0.1.2.3......\}.$$

We denoted by (S, T)-$\mathcal{O}_A(x_0)$.
(ii) *The T-orbit of A in x_0 is the (S, T)-orbit of A in x_0 which $T = S$, we denoted by $\mathcal{O}_A^T(x_0)$.*

2 Common fixed point for (S,T)-Boyd-Wong contraction

Theorem 2.1 *Let (X,F,τ) be a complete PM space which $RanF \subseteq D^+$, and let A,B,S and T be self mappings of X such that the following conditions are satisfied:*
(i) *$AX \subseteq TX$ and $BX \subseteq SX$.*
(ii) *$[A,S]$ and $[B,T]$ are compatible.*
(iii) *S and T are continuous.*
(iv) *There exist a bounded sequence (y_n) compatible with $[A,B,T,S]$ in some $x_0 \in X$.*
If (A,B) is (S,T)-Boyd-Wong contraction, then A,B,S and T have a unique common fixed point.

Proof Let (y_n) the bounded sequence compatible with $[A,B,T,S]$ in some $x_0 \in X$. So there exist a sequence $(x_n) \subset X$ where

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+2} = Sx_{2n+2} = Bx_{2n+1} \quad \text{for} \quad n = 0,1,2.....$$

Since (A,B) is (S,T)-Boyd-Wong contraction, then there exists $\phi \in \Phi$ such that

$$F_{Ax_{2n}Bx_{2n+1}}(t) \geq F_{Sx_{2n}Tx_{2n+1}}(\phi(t)) \quad \textit{and} \quad F_{Ax_{2n+2}Bx_{2n+1}}(t) \geq F_{Sx_{2n+2}Tx_{2n+1}}(\phi(t)).$$

for all $t > 0$. it follows that

$$F_{y_{2n+1}y_{2n+2}}(t) \geq F_{y_{2n}y_{2n+1}}(\phi(t)) \quad \textit{and} \quad F_{y_{2n+2}y_{2n+3}}(t) \geq F_{y_{2n+1}y_{2n+2}}(\phi(t)).$$

Therefore

$$F_{y_{n+1}y_{n+2}}(t) \geq F_{y_ny_{n+1}}(\phi(t)).$$

By induction

$$F_{y_ny_{n+1}}(t) \geq F_{y_0y_1}(\phi^n(t)).$$

Letting $n \to \infty$ in the above inequality and using Lemma 1.10, we get

$$F_{y_ny_{n+1}} \to \epsilon_0.$$

We next prove that (y_n) is a Cauchy sequence. Setting

$$\Lambda = \{y_n : n = 0.1.2......\},$$

and discuss the following cases.

Case (a) n and p are odd or n is even and p is odd, then

$$F_{y_ny_{n+p}}(t) \geq F_{y_0y_p}(\phi^n(t)) \geq D_\Lambda(\phi^n(t)).$$

Hence

$$F_{y_n y_{n+p}} \longrightarrow \epsilon_0 \text{ as } n \to \infty.$$

Case (b) n and $p$are even or n is odd and p is even, then

$$F_{y_n y_{n+p}} \geq \tau(F_{y_n y_{n+1}}, F_{y_{n+1} y_{n+p}}) \geq \tau(F_{y_0 y_1}(\phi^n), F_{y_1 y_p}(\phi^n)).$$

So

$$F_{y_n y_{n+p}} \geq \tau(F_{y_0 y_1}(\phi^n), D_\Lambda(\phi^n)).$$

Consequently

$$F_{y_n y_{n+p}} \longrightarrow \epsilon_0 \text{ as } n \to \infty.$$

Considering all the above cases, we conclude that (y_n) is a Cauchy sequence, hence is convergent in complete PM space (X, F, τ). Let $y \in X$ such that

$$y_n \to y \text{ as } n \to \infty.$$

Consequently (Ax_{2n}), (B_{2n+1}), (Sx_{2n}) and (Tx_{2n+1}) also converges to y, which implies, from Lemma 1.16 that

$$ASx_{2n} \longrightarrow Sy \text{ and } BTx_{2n+1} \longrightarrow Ty.$$

Since (A, B) is (S, T)-Boyd-Wong contraction, we have

$$F_{Ax_{2n}BTx_{2n+1}}(t) \geq F_{Sx_{2n}TTx_{2n+1}}(\phi(t));$$

and

$$F_{ASx_{2n}Bx_{2n+1}}(t) \geq F_{SSx_{2n}Tx_{2n+1}}(\phi(t)).$$

for every $t > 0$ and $n \in \mathbb{N}$. Taking $n \to \infty$, we get

$$F_{yTy}(t) \geq F_{yTy}(\phi(t)) \text{ and } F_{ySy}(t) \geq F_{ySy}(\phi(t))$$

for all $t > 0$. By Lemma 1.17 we concluded that

$$Ty = y \text{ and } Sy = y.$$

Again, using the fact that (A, B) is (S, T)-Boyd-Wong contraction, we obtain

$$F_{AyBx_{2n+1}}(t) \geq F_{yTx_{2n+1}}(\phi(t)) = F_{SyTx_{2n+1}}(\phi(t));$$

and

$$F_{Ax_{2n}By}(t) \geq F_{Sx_{2n}y}(\phi(t)) = F_{Sx_{2n}Ty}(\phi(t)).$$

Taking $n \longrightarrow \infty$, we get

$$Ay = y \text{ and } By = y.$$

Therefore y is a common fixed point of $A,\ B,\ S$ and T.
To prove the uniqueness of the common fixed point, let us suppose that there exists another common fixed point $z \in X$ of $A,\ B,\ S$ and T, then

$$F_{zy}(t) = F_{AzBy}(t) \geq F_{SzTy}(\phi(t)) = F_{zy}(\phi(t))$$

for all $t > 0$.
By Lemma 1.17, we have $y = z$.

Remark 2.2 *Notice that the condition-hypothesis (iv) of Theorem 2.1 is necessary condition of the existence of fixed point as the following Sherwood's example shows*

Example 2.3 [8] *Let G be the distribution function defined by*

$$G(t) = \begin{cases} 0, & t \leq 4 \\ 1 - \frac{1}{n}, & 2^n < t \leq 2^{n+1} \ n > 1. \end{cases}$$

Consider the set $X = \{1, 2, ..., n, ...\}$ and define F on $X \times X$ as follows:

$$F_{nm}(t) = F_{mn}(t) = \begin{cases} 0, & t \leq 0 \\ T_L^m(G(2^n t), G(2^{2n+1}t), ...G(2^{n+m}t)), & t > 0. \end{cases}$$

Then (X, F, τ_{T_L}) is a complete PM space and the mapping $g(n) = n + 1$, satisfying

$$F_{g(n)g(m)}(t) \geq F_{nm}(2t)$$

for all $n, m \in M$ and $t > 0$ (i.e. g is a Boyd-Wong contraction with $\phi(t) = 2t$). But g is fixed point free mapping. Since there does not exist n in X, such that $\mathcal{O}_g(n)$ is bounded.

Remark 2.4 *The condition-hypothesis $RanF \subseteq D^+$ of Theorem 2.1 is necessary condition for the uniqueness of the fixed point as the following Example shows*

Example 2.5 *Consider the set $X = \{p, q\}$ and define F on $X \times X$ as follows:*

$$F_{pq}(t) = \frac{1}{2}(\varepsilon_0 + \varepsilon_\infty)$$

Then (X, F, τ_M) is a complete PM space and the identity mapping id_X on X, satisfying

$$F_{nm}(t) \geq F_{nm}(2t)$$

for all $n, m \in X$ and $t > 0$ (i.e. id_X is a Boyd-Wong contraction with $\phi(t) = 2t$). But id_m has two fixed point . Since $F_{pq} \notin D^+$.

As a direct consequence of Theorem 2.1, if $A = B$ i.e A is a (S, T)-Boyd-Wong contractive mapping we have the following

Corollary 2.6 *Let (X, F, τ) be a complete PM space which $RanF \subseteq D^+$, and let A, S and T are self mappings of X such that the following conditions are satisfied:*
(i) $AX \subseteq TX \cap SX$.
(ii) *S and T are continuous.*
(iii) *$[A, T]$ and $[A, S]$ are compatible.*
(iv) *There exist $x_0 \in X$ such that (S, T)-$\mathcal{O}_A(x_0)$ is bounded.*
If A is (S, T)-Boyd-Wong contraction, then A, S and T have a unique common fixed point.

Before the next Remark, we recall the Definition of a fuzzy metric space

Definition 2.7 *A fuzzy metric space is a triple (X, M, T), where X is an arbitrary nonempty set, T is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following condition:*
(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ *if and only if* $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, z, t + s) \geq T(M(x, y, t), M(y, z, s))$,
(v) $M(x, y, .) : (0, \infty) \to [0, 1]$ *is continuous,*
for all x, y, $z \in X$ and t, $s > 0$.

Remark 2.8 *First note that in the Proof of main result the condition "$F_{pq}(0) = 0$" was not used, then all results mentioned here remains true in fuzzy metric space*

Finally, we give a simple Example to discuss the validity of the hypothesis of Theorem 2.1

Example 2.9 *Let $X = [0, \infty)$. Define $F : X \times X \to \Delta^+$ as follow:*

$$F_{pq}(t) = \varepsilon_0(t - |p - q|).$$

It is easy to check that (M, F, τ_M) is a complete PM space. Let us now consider the mappings A, B, T, $S : M \to M$, $A(x) = 2x$, $B(x) = 4x$, $T(x) = 4x$, $S(x) = 8x$. It is very easy to cheek that $AX = BX = SX = TX = X$, $AS = SA$, $BT = TB$, S and T are continuous, $\{0\}$ is a bounded sequence compatible with [A, B, S, T] in 0, (A, B) is (S, T)-Boyd-Wong contraction with $\phi(t) = 2t$, and consequently by an application of Theorem 2.1 A, B, S and T have 0 as a unique common fixed point

References

[1] D. W. Boyd and J. S. W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. **20**, (1969), 458- 464.

[2] M. Elamrani, A. Mbarki and B. Mehdaoui, Nonliear contractions and Semigroups in General complete probabilistic Metric spaces, PanAmerican Mathematical journal, **11 no. 4**, (2001), 79–87.

[3] Menger. K. Statistical metrics. Proc. Nat. Acad. Sci. **28**, (1942), 535–7.

[4] Schweizer B. and A.Sklar, Probabilistic Metric Spaces, North-Holland Series in Probability and Applied Mathimatics, **5**, (1983).

[5] SchweizerB. and A.Sklar, Statistical metric spaces. Pacific J. Math. **10** (1960), 313–34.

[6] Sehgal V.M and Bharucha-Reid A.T, Fixed Point theorems of contraction mappings in probabilistic metric spaces, Math. Systemes Theory, **6**, (1972),97-102.

[7] Seong Hoon Cho, On common fixed points in fuzzy metric spaces, International Mathematical Forum, **1 no. 9-12** , (2006), 471–479.

[8] H. Sherwood, Complete probabilistic metric space, Z. Wahrscheinlichkeitstheorie verw Geb **22**, (1971); 117-28.

[9] Stevens. S, Measurement psychophisics and untility in measurement Definition and theory, C. west Churchman and P-Rtoosh eds., (1959), 18-63.

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