Some Coupled Common Fixed Point Theorems for a Pair of Mappings Satisfying a Contractive Condition of Rational Type without Monotonicity

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Abstract. The purpose of this paper is to establish some coupled coincidence point theorems for a pair of mappings without mixed monotone property satisfying a contractive condition of rational type in the framework of partially ordered metric spaces. Also, we present a result on the existence and uniqueness of coupled common fixed points. The results presented in the paper generalize and extend several well-known results in the literature.

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1 Introduction and Preliminaries

Fixed point theory is one of the famous and traditional theories in mathematics and has a large number of applications. The Banach contraction mapping is one of the pivotal results of analysis. It is a very popular tool for solving existence problems in many different fields of mathematics. There are a lot of generalizations of the Banach contraction principle in the literature. Ran and Reurings [13] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. While Nieto and Rodríguez-López [12] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [1] introduced the concept of mixed monotone mappings and obtained some coupled fixed point results. Also, they applied their results on a first order differential equation with periodic boundary conditions. Recently, many researchers have obtained fixed point, common fixed point, coupled fixed point and coupled common fixed point results in cone metric spaces, partially ordered metric spaces and others (see [1]-[13] and references cited there in).

Definition 1.1. Let \((X, d)\) be a metric space and \(F: X \times X \to X\) and \(g: X \to X\), \(F\) and \(g\) is said to commute if \(F(gx, gy) = g(F(x, y))\), for all \(x, y \in X\).

Definition 1.2. Let \((X, d)\) be a metric space and let \(g: X \to X\), \(F: X \times X \to X\). The mappings \(g\) and \(F\) are said to be compatible if \(\lim_{n \to \infty} d(gsx_n, y_n, F(gsx_n, gy_n)) = 0\) and \(\lim_{n \to \infty} d(gsF(x_n, y_n), F(gsx_n, gy_n)) = 0\), hold whenever \(\{x_n\}\) and \(\{y_n\}\) are sequences in \(X\) such that \(\lim_{n \to \infty} F(x_n, y_n) = \lim_{n \to \infty} gsx_n\) and \(\lim_{n \to \infty} F(y_n, x_n) = \lim_{n \to \infty} gy_n\).

Definition 1.3. Let \((X, \preceq)\) be a partially ordered set and \(F: X \to X\). The mapping \(F\) is said to be non-decreasing if for \(x, y \in X\), \(x \preceq y\) implies \(F(x) \preceq F(y)\) and non-increasing if for \(x, y \in X\), \(x \preceq y\) implies \(F(x) \succeq F(y)\).

Definition 1.4. Let \((X, \preceq)\) be a partially ordered set and \(F: X \times X \to X\) and \(g: X \to X\). The mapping \(F\) is said to have the mixed \(g\)-monotone property if \(F(x, y)\) is monotone \(g\)-non-decreasing in \(x\) and monotone \(g\)-non-increasing in \(y\), that is, for any \(x, y \in X\),

\[
x_1, x_2 \in X, gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),
\]

and

\[
y_1, y_2 \in X, gy_1 \preceq gy_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).
\]
If \( g = \text{identity mapping} \) in Definition 1.4, then the mapping \( F \) is said to have the \textit{mixed monotone property}.

Recently, Doric et al. [9] showed that a mixed monotone property in coupled fixed point results for mappings in ordered metric spaces can be replaced by another property which is often easy to check. In particular, it is automatically satisfied in the case of a totally ordered space, the case which is important in applications. Hence, these results can be applied in a much wider class of problems. The purpose of this paper is to establish some coupled coincidence point results in partially ordered metric spaces for a pair of mappings without mixed monotone property satisfying a contractive condition of rational type. Also, we present a result on the existence and uniqueness of coupled common fixed points.

If elements \( x, y \) of a partially ordered set \((X, \preceq)\) are comparable (i.e. \( x \preceq y \) or \( y \preceq x \) holds) we will write \( x \preceq y \). Let \( g : X \to X \) and \( F : X \times X \to X \). We will consider the following condition:

\[
\text{if } x, y, u, v \in X \text{ are such that } gx \preceq F(x, y) = gu \text{ then } F(x, y) \preceq F(u, v).
\]

If \( g \) is an identity mapping, for all \( x, y, v \) if \( x \preceq F(x, y) \) then \( F(x, y) \preceq F(F(x, y), v) \).

Doric et al. [9] give some examples that these conditions may be satisfied when \( F \) does not have the \( g \)-mixed monotone property.

**Definition 1.5.** An element \((x, y) \in X \times X\) is called a \textit{coupled coincidence point} of the mappings \( F : X \times X \to X \) and \( g : X \to X \) if \( F(x, y) = gx \), and \( F(y, x) = gy \).

If \( g = \text{identity mapping} \) in Definition 1.5, then \((x, y) \in X \times X\) is called a \textit{coupled fixed point}.

## 2 Main Results

**Theorem 2.1.** Let \((X, d, \preceq)\) be a complete partially ordered metric space and let \( F : X \times X \to X \) and \( g : X \to X \). Suppose that the following hold:

(i) \( g \) is continuous and \( g(X) \) is closed;

(ii) \( F(X \times X) \subseteq g(X) \) and \( g \) and \( F \) are compatible;

(iii) for all \( x, y, u, v \in X \), if \( g(x) \preceq F(x, y) = gu \), then \( F(x, y) \preceq F(u, v) \);

(iv) there exist \( x_0, y_0 \in X \) such that \( gx_0 \preceq F(x_0, y_0) \) and \( gy_0 \preceq F(y_0, x_0) \),

(v) there exists \( \alpha \in [0, 1) \) such that for all \( x, y, u, v \in X \) with \( gx \preceq gu \) and \( gy \preceq gv \),
satisfies,
\[
    d(F(x, y), F(u, v)) \leq \alpha \max \{d(gx, gu), d(gy, gv), \frac{d(gx, F(x, y))d(gu, F(u, v))}{d(gx, gu)}, \frac{d(gx, F(u, v))d(gu, F(x, y))}{d(gx, gu)}, \frac{d(gy, F(v, u))d(gv, F(y, x))}{d(gy, gv)}\}.
\]

(vi) \(F\) is continuous.

Then there exist \(x, y \in X\) such that \(F(x, y) = g(x)\) and \(gy = F(y, x)\), that is, \(F\) and \(g\) have a coupled coincidence point \((x, y) \in X \times X\).

**Proof.** Using condition (ii) and (iv), construct sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) satisfying \(gx_n = F(x_{n-1}, y_{n-1})\) and \(gy_n = F(y_{n-1}, x_{n-1})\) for \(n = 1, 2, \ldots\)

By (iv), \(gx_0 \preceq F(x_0, y_0) = gx_1\) and condition (iii) implies that \(gx_1 = F(x_0, y_0) \preceq F(x_1, y_1) = gx_2\). Proceeding by induction, we get that \(gx_{n-1} \preceq gx_n\), and similarly, \(gy_{n-1} \preceq gy_n\) for each \(n \in \mathbb{N}\).

Now from the contractive condition (2.1), we have
\[
    d(gx_{n+1}, gx_n) = d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \leq \alpha \max \{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), \frac{d(gx_n, F(x_n, y_n))d(gx_{n-1}, F(x_{n-1}, y_{n-1}))}{d(gx_n, gx_{n-1})}, \frac{d(gx_n, F(x_{n-1}, y_{n-1}))d(gx_{n-1}, F(x_{n-1}, y_{n-1}))}{d(gx_n, gx_{n-1})}, \frac{d(gy_n, F(y_{n-1}, x_{n-1}))d(gy_{n-1}, F(y_{n-1}, x_{n-1}))}{d(gy_n, gy_{n-1})}, \frac{d(gy_n, F(y_{n-1}, x_{n-1}))d(gy_{n-1}, F(y_{n-1}, x_{n-1}))}{d(gy_n, gy_{n-1})}\}.
\]

which implies that \(d(gx_{n+1}, gx_n) \leq \alpha \max \{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}\). Similarly, we have \(d(gy_{n+1}, gy_n) \leq \alpha \max \{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}\). Set \(\varrho_n := \max \{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}\). Hence \(\max \{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\} \leq \alpha \varrho_{n-1}\). By induction we get that \(\max \{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\} \leq \alpha^n \varrho_0\). It easily follows that for each \(m, n \in \mathbb{N}, m < n\), we have
\[
    d(gx_m, gx_n) \leq \frac{\alpha^m}{1 - \alpha} \varrho_0,
\]
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and

\[ d(gy_m, gy_n) \leq \frac{\alpha^m}{1 - \alpha^0}. \]

Therefore, \( \{gx_n\} \) and \( \{gy_n\} \) are Cauchy sequences and since \( g(X) \) is closed in a complete metric space, there exist \( x, y \in g(X) \) such that

\[ \lim_{n \to \infty} gx_n = \lim_{n \to \infty} F(x_{n-1}, y_{n-1}) = x \quad \text{and} \quad \lim_{n \to \infty} gy_n = \lim_{n \to \infty} F(y_{n-1}, x_{n-1}) = y. \]

Compatibility of \( F \) and \( g \) implies that \( \lim_{n \to \infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0 \) and \( \lim_{n \to \infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0. \) As \( F \) is continuous. Then \( F(gx_n, gy_n) \to F(x, y) \) and \( F(gy_n, gx_n) \to F(y, x) \). Using triangle inequality, we get

\[ d(gx, F(gx_n, gy_n)) \leq d(gx, g(F(x_n, y_n))) + d(g(F(x_n, y_n)), F(gx_n, gy_n)). \]

By taking limit \( n \to \infty \), and using continuity of \( F \) and \( g \), we have \( d(gx, F(x, y)) = 0 \) i.e. \( gx = F(x, y) \) and in similar way, we have \( gy = F(y, x) \). Thus \( F \) and \( g \) have a coupled coincidence point.

If \( g \) is an identity mapping in above theorem we have the following result.

**Corollary 2.2.** Let \( (X, d, \leq) \) be a complete partially ordered metric space and let \( F : X \times X \to X \). Suppose that the following hold:

(i) for all \( x, y, v \in X \), if \( x \leq F(x, y) \), then \( F(x, y) \leq F(F(x, y), v) \);

(ii) there exist \( x_0, y_0 \in X \) such that \( x_0 \leq F(x_0, y_0) \) and \( y_0 \leq F(y_0, x_0) \).

(iii) there exists \( \alpha \in [0, 1) \) such that for all \( x, y, u, v \in X \) with \( x \leq u \) and \( y \succeq v \), satisfies,

\[
    d(F(x, y), F(u, v)) \leq \alpha \max\{d(x, u), d(y, v), \frac{d(x, F(x, y))d(u, F(u, v))}{d(x, u)}, \frac{d(x, F(u, v))d(u, F(x, y))}{d(x, u)}, \frac{d(y, F(y, x))d(v, F(v, u))}{d(y, v)}, \frac{d(y, F(v, u))d(v, F(y, x))}{d(y, v)} \},
\]

(iv) \( F \) is continuous.

Then there exist \( x, y \in X \) such that \( F(x, y) = x \) and \( y = F(y, x) \), that is, \( F \) has a coupled fixed point \( (x, y) \in X \times X \).

Now, we shall prove the existence and uniqueness of a coupled common fixed point. Note that, if \( (X, \leq) \) is a partially ordered set, then we endow the product space \( X \times X \) with the following partial order relation:

\[ (x, y), (u, v) \in X \times X, (u, v) \preceq (x, y) \iff x \preceq u, y \succeq v. \]
Theorem 2.3. In addition to hypotheses of Theorem 2.1, suppose that for every \((x, y), (z, t) \in X \times X\), there exists \((u, v) \in X \times X\) such that \((F(u, v), F(v, u))\) is comparable to both \((F(x, y), F(y, x))\) and \((F(z, t), F(t, z))\). Then \(F\) and \(g\) have a unique coupled common fixed point, that is, there exists a unique \((p, q) \in X \times X\) such that \(p = gp = F(p, q)\) and \(q = gq = F(q, p)\).

Proof. From Theorem 2.1, the set of coupled coincidence points of \(F\) and \(g\) is non-empty. Suppose that \((x, y)\) and \((z, t)\) are coupled coincidence points of \(F\) and \(g\), that is, \(gx = F(x, y), gy = F(y, x), gz = F(z, t)\) and \(gt = F(t, z)\). We shall show that \(gx = gz\) and \(gy = gt\). By the assumption, there exists \((u, v) \in X \times X\) such that \((F(u, v), F(v, u))\) is comparable to \((F(x, y), F(y, x))\) and \((F(z, t), F(t, z))\). Put \(u_0 = u, v_0 = v\) and choose \(u_1, v_1 \in X\) so that \(gu_1 = F(u_0, v_0)\) and \(gv_1 = F(v_0, u_0)\). Then similarly as in the proof of Theorem 2.1, we can inductively define sequences \(\{gu_n\}, \{gv_n\}\) as \(gu_{n+1} = F(u_n, v_n)\) and \(gv_{n+1} = F(v_n, u_n)\) for all \(n\). Further, set \(x_0 = x, y_0 = y, z_0 = z, t_0 = t\) and on the same way define the sequences \(\{gx_n\}, \{gy_n\}, \{gz_n\}\) and \(\{gt_n\}\). Then as in Theorem 2.1, we can show that \(gx_n \to gx = F(x, y), gy_n \to gy = F(y, x), gz_n \to gz = F(z, t), gt_n \to gt = F(t, z),\) for all \(n \geq 1\). Since \((F(x, y), F(y, x)) = (gx, gy)\) and \((F(u, v), F(v, u)) = (gu_1, gv_1)\) are comparable, then \(gx \preceq gy\) and \(gy \preceq gx\) and in a similar way, we have \(gu_n = F(u_{n-1}, v_{n-1}) \preceq F(x, y) = gx\) and \(gv_n = F(v_{n-1}, u_{n-1}) \preceq F(y, x) = gy\). Thus from (2.1), we have

\[
\begin{align*}
d(gx, gu_{n+1}) &= d(F(x, y), F(u_n, v_n)) \\
&\leq \alpha \max\{d(gx, gu_n), d(gy, gv_n)\} \frac{d(gx, F(x, y))d(gu_n, F(u_n, v_n))}{d(gx, gu_n)} \\
&\quad \frac{d(gx, F(u_n, v_n))d(gu_n, F(x, y))}{d(gx, gu_n)} \frac{d(gy, F(y, x))d(gv_n, F(v_n, u_n))}{d(gy, gv_n)} \\
&\quad \frac{d(gy, F(u_n, v_n))d(gv_n, F(y, x))}{d(gy, gv_n)} \\
&= \alpha \max\{d(gx, gu_n), d(gy, gv_n), d(gx, gu_{n+1}), d(gy, gv_{n+1})\}.
\end{align*}
\]

(2.4)

Similarly, we can prove that

\[
d(gy, gv_{n+1}) \leq \alpha \max\{d(gx, gu_n), d(gy, gv_n), d(gx, gu_{n+1}), d(gy, gv_{n+1})\}.
\]

Hence

\[
\max\{d(gx, gu_{n+1}), d(gy, gv_{n+1})\} \leq \alpha \max\{d(gx, gu_n), d(gy, gv_n)\}
\]

and by induction

\[
\max\{d(gx, gu_{n+1}), d(gy, gv_{n+1})\} \leq \alpha^n \max\{d(gx, gu_1), d(gy, gv_1)\}
\]
On taking limit, \( n \to \infty \), we get \( \lim_{n \to \infty} d(gx, gu_{n+1}) = 0 \) and \( \lim_{n \to \infty} d(gy, gv_{n+1}) = 0 \). Similarly, we can prove that \( \lim_{n \to \infty} d(gz, gu_{n+1}) = 0 = \lim_{n \to \infty} d(gt, gv_{n+1}) \). Finally, we have \( d(gx, gz) \leq d(gx, gu_{n}) + d(gu_{n}, gz) \) and \( d(gy, gt) \leq d(gy, gv_{n}) + d(gv_{n}, gt) \). Taking \( n \to \infty \) in these inequalities, we get \( d(gx, gz) = 0 = d(gy, gt) \), that is \( gx = gz \) and \( gy = gt \). Denote \( gx = p \) and \( gy = q \). we have that \( gp = g(F(x, y)) \) and \( gq = g(F(y, x)) \). By the definition of sequences \( \{x_n\} \) and \( \{y_n\} \), we have \( gx_n = F(x_{n-1}, y_{n-1}) \) and \( gy_n = F(y_{n-1}, x_{n-1}) \), and so \( gx_n \to F(x, y) \) and \( gy_n \to F(y, x) \). Compatibility of \( F \) and \( g \) implies that \( \lim_{n \to \infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) \to 0 \) i.e. \( g(F(x, y)) = F(gx, gy) \). Then \( gp = F(p, q) \) and similarly, \( gq = F(q, p) \). Thus \( (p, q) \) is a coupled coincidence point. Thus, it follows \( gp = gx \) and \( gq = gy \), that is, \( gp = p \) and \( gq = q \). Hence \( p = gp = F(p, q) \) and \( q = gq = F(q, p) \). Therefore, \( (p, q) \) is a coupled common fixed point of \( F \) and \( g \). To prove the uniqueness, assume that \( (r, s) \) is another coupled common fixed point. Then as above, we have \( r = gr = gp = p \) and \( s = gs = gq = q \). Hence we get the result.

If \( g \) is an identity mapping in above theorem we have the following result.

**Corollary 2.4.** In addition to hypotheses of Corollary (2.2), suppose that for every \( (x, y), (z, t) \in X \times X \), there exists \( (u, v) \in X \times X \) such that \( (F(u, v), F(v, u)) \) is comparable to both \( (F(x, y), F(y, x)) \) and \( (F(z, t), F(t, z)) \). Then the coupled fixed point of \( F \) is unique.

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