On Some Identities for k-Jacobsthal Numbers

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Abstract

The aim of this paper is to obtain Binet formula for k-Jacobsthal numbers. And also with the help of Binet formula we obtain some properties for the k-Jacobsthal numbers.

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1. Introduction

In recent year, Fibonacci numbers and their generalization have many interesting properties and application to almost every field of science and art. Koshy [12] has devoted nearly 700 pages to the properties of Fibonacci and Lucas number, with scarcely a mention of general two term recurrences. For further more links can be seen in [13], [8], [11].

In [9] Falcon and Plaza found general k-Fibonacci numbers and obtained many properties of these numbers directly from elementary matrix algebra. Also, In [10] Falcon and Plaza defined k-hyperbolic function. In [5] Bolat and Köse obtain identities including generating function and divisibility properties for k- Fibonacci number. In [6] Koken and Bozkurt deduce some properties and Binet like formula for the Jacobsthal number by matrix method. In this paper, we present the k-Jacobsthal number in an explicit way, and many properties are proved by easy arguments for the k-Jacobsthal number.

2. The k-Jacobsthal Number and Properties

For any positive real number k, the k-Jacobsthal sequence say \(\{J_{k,n}\}_{n\in\mathbb{N}}\) is defined recurrently by

\[ J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}; \text{ for } n \geq 1 \]  

(2.1)
With initial condition \( J_{k,0} = 0, \ J_{k,1} = 1 \) \hfill (2.2)

### 2.1 Explicit formula for the general term of the \( k \)-Jacobsthal sequence

Binet’s formulas are well known in [4,12]. In our case, Binet’s formula allows us to express the \( k \)-Jacobsthal numbers in function of the roots \( r_1 \) and \( r_2 \) of the following characteristic equation, associated to the recurrence relation (2.1).

\[
 r^2 = kr + 2 \quad \text{(2.3)}
\]

**Proposition 2.1 (Binet’s formula)**

The \( n \)th \( k \)-Jacobsthal number is given by

\[
 J_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \quad \text{(2.4)}
\]

where \( r_1, r_2 \) are the roots of the characteristic equation (2.3) and \( r_1 > r_2 \)

**Proof:** The roots of the characteristic equation (2.3) are

\[
 r_1, r_2 = \frac{k \pm \sqrt{k^2 + 8}}{2}, \quad r_2 = \frac{k - \sqrt{k^2 + 8}}{2}
\]

Note that, since \( k > 0 \), then \( r_2 < 0 < r_1 \) and \( |r_2| < |r_1| \)

Therefore, the general term of the \( k \)-Jacobsthal sequence may be expressed in the form:

\[ J_{k,n} = c_1 r_1^n + c_2 r_2^n \quad \text{for some coefficients} \quad c_1 \text{ and } c_2. \]

Giving to \( n \) the values \( n = 0 \) and \( n = 1 \) it is obtained \( c_1 = \frac{1}{r_1 - r_2} = -c_2 \), and therefore \( J_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \).

**Proposition 2.2 (Catalan’s identity)**

\[
 J_{k,n} J_{k,n+r} - J_{k,n+r}^2 = (-1)^{n+1-r} J_{k,r}^2 2^{n-r} \quad \text{(2.5)}
\]

**Proof:** By using Eq. (2.4) in the left hand side (LHS) of Eq. (2.5), and taking into account that \( r_1 r_2 = -1 \) it is obtained

\[
 J_{k,n}, J_{k,n+r} - J_{k,n+r}^2 = \frac{r_1^{n-r} - r_2^{n-r} r_1^{n+r} - r_2^{n+r}}{r_1 - r_2} - \left( \frac{r_1^n - r_2^n}{r_1 - r_2} \right)^2
 = \frac{(-1)^{n+1} (2)^n (r_2^{2r} + r_1^{2r})}{(r_1 - r_2)^2 (r_1 r_2)^{-2}} - 2
 = (-1)^{n+1-r} (2)^{n-r} J_{k,r}^2
\]
Note that for \( r = 1 \), Eq. (2.5) gives Cassini’s identity for the k-Jacobsthal sequence

\[
J_{k,n-1}J_{k,n+1} - J_{k,n}^2 = (-1)^n (2)^{n-r}
\]  

(2.6)

**Proposition 2.3 (D’ocagne’s identity)**

If \( m > n \) then

\[
J_{k,m}J_{k,n+1} - J_{k,m+1}J_{k,n} = (-2)^m J_{k,m-n}
\]  

(2.7)

**Proof:** By using Eq. (2.4)

\[
J_{k,n}J_{k,n+1} - J_{k,m+1}J_{k,m} = \frac{r_1^n - r_2^n}{r_1 - r_2} \frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2} - \frac{r_1^{m+1} - r_2^{m+1}}{r_1 - r_2} \frac{r_1^m - r_2^m}{r_1 - r_2}
\]

\[
= (r_1r_2)^n \left( \frac{r_1^{m-n} - r_2^{m-n}}{r_1 - r_2} \right)
\]

\[
= (-2)^m J_{k,m-n}
\]

2.2 Another explicit expression for calculating the general term of the k-Jacobsthal sequence is given by the following preposition-

**Proposition 2.4**

\[
J_{k,n} = \frac{1}{2^n-1} \sum_{i=0}^{n-1} \left( \frac{n}{2i+1} \right) k^{n-1-2i} \left( k^2 + 8 \right)
\]

(2.8)

where \( \lfloor a \rfloor \) is the floor function of \( a \), that is \( \lfloor a \rfloor = \sup \{ n \in \mathbb{N} ; n \leq a \} \) and says the integer part of \( a \), for \( a \geq 0 \).

**Proof:** By using the values of \( r_1 \) and \( r_2 \) obtained in Eq. (2.4), we get

\[
J_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} = \frac{1}{\sqrt{k^2+8}} \left[ \left( \frac{k + \sqrt{k^2+8}}{2} \right)^n - \left( \frac{k - \sqrt{k^2+8}}{2} \right)^n \right]
\]

From where, by developing the nth powers, it follows:

\[
= \frac{1}{\sqrt{k^2+8}} \left[ \sum_{i=0}^{n-1} \left( \frac{n}{2i+1} \right) \left( \frac{\sqrt{k^2+8}}{k} \right)^i \left( \frac{k^3}{k^3} \right)^i \right]
\]

\[
= \frac{1}{2^{n-1}} \sum_{i=0}^{n-1} \left( \frac{n}{2i+1} \right) k^{n-1-2i} \left( k^2 + 8 \right)
\]
2.3 Limit of the quotient of two consecutive terms is equal to the positive root of the corresponding characteristic equation

**Proposition 2.5**

\[
\lim_{n \to \infty} \frac{J_{k,n}}{J_{k,n-1}} = r_1
\]  \hspace{1cm} (2.9)

**Proof.** By using Eq. (2.4)

\[
\lim_{n \to \infty} \frac{J_{k,n}}{J_{k,n-1}} = \lim_{n \to \infty} \frac{r_1^n - r_2^n}{r_1^{n-1} - r_2^{n-1}} = \lim_{n \to \infty} \frac{1 - \left(\frac{r_2}{r_1}\right)^n}{\sqrt[n]{r_1^n} - \left(\frac{r_1}{r_2}\right)^n \frac{1}{r_2}}
\]

and taking into account that \(\lim_{n \to \infty} \left(\frac{r_2}{r_1}\right)^n = 0\)

since \(|r_2| < 1\), Eq. (2.9) is obtained.

3. Generating functions for the k-Jacobsthal sequences

In this section, the generating functions for the k-Jacobsthal sequences are given. As a result, k-Jacobsthal sequences are seen as the coefficients of the power series of the corresponding generating function.

Let us suppose that the Jacobsthal numbers of order k are the coefficients of a potential series centered at the origin, and let us consider the corresponding analytic function \(j_k(x)\) the function defined in such a way is called the generating function of the k-Jacobsthal numbers.

So,

\[
j_k(x) = J_{k,0} + J_{k,1}x + J_{k,2}x^2 + \ldots + J_{k,n}x^n
\]

And then,

\[
kxj_k(x) = kJ_{k,0}x + kJ_{k,1}x^2 + kJ_{k,2}x^3 + \ldots + kJ_{k,n}x^{n+1}
\]

\[2x^2 j_k(x) = 2J_{k,0}x^2 + 2J_{k,1}x^3 + 2J_{k,2}x^4 + \ldots + 2J_{k,n}x^{n+2}\]
From where, since $J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}$, $J_{k,0} = 0$ and $J_{k,1} = 1$, it is obtained

$$\left(1-kx-2x^2\right) j_k(x) = x$$

So the generating function for k-Jacobsthal sequence $\{J_{k,n}\}_{n=0}^\infty$ is $j_k(x) = \frac{x}{1-kx-2x^2}$.

References


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