More on the Subnormal 2-Variable Weighted Shifts and Linear Recursive Relations

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Abstract
The main purpose of this paper aims at studying the subnormal 2-variable weighted shifts; where the weights are associated to a family of sequences defined by linear recursive relations. Some results are provided, and others properties for 1-variable weighted shift are also extended. Finally, an elementary computation gives rise to a concrete criterion of the subnormality for the 2-variable weighted shift operator.

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1 Introduction
The moment problem, largely studied in the literature, arises in various fields of mathematics, particularly its results and techniques are important in the operator theory (see [1], [2], [3], [7], [13] for example). Specially, it is well known that these results are very useful for characterizing the subnormality (see [4], [6], [7], [8] for instance). Recall that the classical $K$-moment problem
for a sequence \( \{\gamma_j\}_{0 \leq j \leq p} \) \((p \leq +\infty)\), consists of finding a positive Borel measure \( \mu \) such that,

\[
\gamma_j = \int_{K} t^j d\mu(t) \quad \text{for every } j \ (0 \leq j \leq p) \quad \text{and } \text{supp}(\mu) \subset K, \quad (1)
\]

where \( K \) is a closed subset of \( \mathbb{R} \). The problem \( (1) \) is called full if \( p = +\infty \) (see for example \([1], [2], [13]\)) and truncated if \( p < +\infty \) (see for instance \([5], [7], [8]\)). A positive Borelean measure \( \mu \) solution of \( (1) \) is called a representing measure of \( \{\gamma_j\}_{0 \leq j \leq p} \). Consider the usual separable Hilbert space \( \mathcal{H} = l^2(\mathbb{Z}) \) and \( \{e_n\}_{n \in \mathbb{Z}_+} \) its orthonormal basis, and let \( \alpha = \{\alpha_n\}_{n \geq 0} \) be a bounded sequence of positive real numbers (called weights). The bounded linear operator on \( \mathcal{H} \) defined by \( W_\alpha e_n = \alpha_n e_{n+1} \) is called the unilateral weighted shift with weight sequence \( \alpha \). The moments of \( \alpha \) are given by \( \gamma_0 = 1 \) and \( \gamma_k \equiv \gamma_k(\alpha) := \alpha_0^2 \alpha_1^2 \cdots \alpha_{k-1}^2 \) if \( k \geq 1 \). It is well known that \( W_\alpha \) can never be normal, and by Berger’s Theorem, \( W_\alpha \) is a subnormal operator, if and only if, there exists a nonnegative Borelean measure \( \mu \) (called Berger measure), which is a representing measure of \( \{\gamma_n\}_{n \geq 0} \) such that \( \supp(\mu) \subset [0, \|W_\alpha\|^2] \) and \( \gamma_n = \int_0^{|W_\alpha|^2} t^n d\mu(t) \) for all \( n \geq 0 \), where \( \|W_\alpha\|^2 = \sup_{n \geq 0} \alpha_n \) (see \([8], [12]\), for example). Hence, the moment problem \((1)\) and subnormality are closely related.

The truncated \( K \)-moment problem \((1)\) is connected to the subnormality via the subnormal completion problem (SCP for short). That is, the SCP for 1-variable weighted shift consists in finding necessary and sufficient conditions on given sequence of positive real numbers \( \{\alpha_n\}_{0 \leq n \leq p} \) such that there exists a subnormal weighted shift whose initial weights are \( \{\alpha_n\}_{0 \leq n \leq p} \) (see \([8]\)). In particular, a subnormal completion criterion was given by Curto-Fialkow in Theorem 3.5 of \([8]\). In various studies on the SCP in 1-variable, the recursive

ness plays a central role in the explicit calculation of the subnormal completion weighted shifts (see \([4], [8]\)). In fact, the sequence \( \gamma \equiv \{\gamma_j\}_{0 \leq j \leq p} \) satisfies the following recursive relation,

\[
\gamma_{n+1} = a_0 \gamma_n + a_1 \gamma_{n-1} + \cdots + a_{r-1} \gamma_{n-r+1} \quad \text{for every } n \geq r, \quad (2)
\]

where \( a_0, a_1, \ldots, a_{r-1} \ (r \geq 2) \) are some fixed numbers with \( a_{r-1} \neq 0 \) and \( \gamma_0, \gamma_1, \ldots, \gamma_{r-1} \) are the initial conditions. Sequences \((2)\) are known in the literature as Fibonacci sequences of order \( r \). When \( \gamma \) satisfies \((2)\), we say that the associated weighted shift \( \alpha \) is recursively generated weighted shift. Involving the sequence \((2)\), it was established that if the roots of the characteristic polynomial \( P(\gamma)(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1} z - a_r \) of a positive sequence \( \gamma \), are simple and positive, then \( W_\alpha \) is subnormal \( \iff \) \( W_\alpha \) is \( k \)-hyponormal \( \iff \) there exists \( k_0 \leq r - 1 \) such that \( W_\alpha \) is \( k_0 \)-hyponormal (see Proposition 3.3 in \([4]\)).

Let \( K = l^2(\mathbb{Z}^2_+) \) be the Hilbert space of square-summable complex sequences indexed by \( \mathbb{Z}^2_+ \) and consider the double-indexed positive bounded sequences
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\{\alpha_k\}, \{\beta_k\} \in l^\infty(\mathbb{Z}_+^2), \text{ } k = (k_1, k_2) \in \mathbb{Z}_+^2. \text{ We define the 2-variable weighted shift } T = (T_1, T_2) \text{ by } T_1e_k := \alpha_k e_{k+1} \text{ and } T_2e_k := \beta_k e_{k+2}, \text{ where } \epsilon_1 = (1,0), \epsilon_2 = (0,1), \text{ satisfying the condition}

\begin{align*}
T_1T_2 = T_2T_2 \iff \beta_k e_{k+\epsilon_1} = \alpha_k e_{k+\epsilon_2} \hat{\beta}_k.
\end{align*}

Equation (3) is called the commutativity condition. The moment of order \( k = (k_1, k_2) \in \mathbb{Z}_+^2 \) for a pair \( \gamma = (\alpha, \beta) \) satisfying (3), is defined by

\[ \gamma_k \equiv \gamma_k(\alpha, \beta) := \begin{cases} 1 & \text{if } k = (0, 0); \\ \alpha_{(0,0)}^2 \cdot \alpha_{(1,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0; \\ \beta_{(0,0)}^2 \cdot \beta_{(0,1)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1; \\ \alpha_{(0,0)}^2 \cdot \alpha_{(1,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & . \end{cases} \]

And the characterization of subnormality for weighted shifts due to Berger (see [8]), and also to Gellar-Wallen (see [12]), can be formulated as follows. The weighted shift operator \( T = (T_1, T_2) \) is subnormal, if and only if, there is a regular Borel probability measure \( \mu \) defined on the 2-dimensional rectangle \( K = [0, a_1] \times [0, a_2] \) \( (a_i := ||T_i||) \) such that

\[ \gamma_k = \int \int_K x^{k_1}y^{k_2}d\mu(x,y) \text{ for every } k \in \mathbb{Z}_+^2. \]

The 2-variable SCP consists, for a given \( m \geq 0 \), to find necessary and sufficient conditions to guarantee the existence of a subnormal 2-variable weighted shift whose initial weights are \( \gamma_m = \{(\alpha_k, \beta_k)\}_{|k| \leq m} \) satisfying (3) for all \( |k| = k_1 + k_2 \leq m \), with \( \alpha_k, \beta_k \) positive numbers (see [10]).

The main goal in this paper is to present some results on the subnormal 2-variable weighted shifts, using the Fibonacci recursiveness approach. More precisely, since for the subnormality in one variable, the explicit calculation requires the recursively generated weighted shifts (see for instance the result of Proposition 3.3 in [4]), we aim to stretch this idea in the 2-variable weighted shifts emanated from the recursive sequences (2). Recall that, the aforementioned approach allowed us to establish the important results for the truncated complex moment sequence in [3], as well as for \( K \)-moment problem for recursive sequence of order \( \infty \) in [5]. Even better, in the present study, it permits us to provide elementary proofs for some primordially results established here below.

The material of this paper is organized as follows. In section 2, we first manage to extend some properties established on the recursively generated weighted shifts for one variable to the 2-variable case, and we later apply them to weak the flatness’s conditions for this study. Moreover, the characterization of subnormality for 2-variable weighted shifts expressed in terms of recursive relations (2) are provided. The \( k \)-hyponormality is deeply related to the positivity of some matrices, for insuring the characterization of subnormality for 2-variable weighted shifts. In Section 3, combining the Hankel matrix associated to the sequence \( \{\gamma_n\}_{n \geq 0} \) with some results of Curto-Fialkow theory and
properties of Section 2, we arrive to establish others characterizations of the subnormality for 2-variable weighted shift. We conclude this section by supplying a concrete criterion typifying the subnormality for the 2-variable weighted, when the shifts \( \{\alpha_k\}_k \) and \( \{\beta_k\}_k \) (\( k \in \mathbb{Z}_+^2 \)) are expressed in terms of sequences (2) of order \( r = 2 \).

2 The subnormal 2-variable weighted shifts and recursiveness

It is known that the notion of recursiveness, associated to a sequence \( \gamma \equiv (\gamma_{ij})_{0 \leq i+j \leq 2n} \), is useful in the Curto-Fialkow’s theory, for insuring the existence of representing measures. Additionally, it was shown that every weighted shift is a norm-limit of recursively generated weighted shifts. For further information, refer to [4], [7] and [8] for instance.

We emphasize here on the fact that Curto-Fialkow’s theory has been of great importance in [3], [4], [5] and [11], where a crucial bridge between the truncated moment problem and the linear generalized Fibonacci relations has been established. More than that, the closed relationship between recursively generated weighted shifts and the linear Fibonacci relation is triggered. Our technique for studying recursively generated weighted shifts is also released from the Fibonacci sequence properties, and it has served in [3] as an important tool for the truncated complex moment sequence. In the present Section, as a first step we review some properties on the recursively generated weighted shifts for 1-variable, and we stretch the results to the 2-variable case. To this end, we need to recall the following result in [3], permitting to characterize the existence of generating measure for a sequence (2).

**Proposition 2.1** Let \( V = \{\gamma_n\}_{n \geq 0} \) be a sequence (2). Then \( \{\gamma_n\}_{n \geq 0} \) admits a generating measure \( \mu \), if and only if, \( P_{\gamma} \) has distinct real roots, where \( P_{\gamma} \) stands for the characteristic polynomial associated to \( \{\gamma_n\}_{n \geq 0} \). Moreover, \( \text{Supp}(\mu) = Z(P_{\gamma}) \).

Let \( T = (T_1, T_2) \) be the 2-variable weighted shift defined by the double-indexed positive bounded sequences \( \{\alpha_k\}, \{\beta_k\} \in l^\infty(\mathbb{Z}_+^2), k = (k_1, k_2) \in \mathbb{Z}_+^2 \), satisfying the commutativity condition (3). We recall the following definition on the flatness of the 2-variable case.

**Definition 2.2** A 2-variable weighted shift \( T = (T_1, T_2) \) is horizontally flat (resp. vertically flat) if \( \alpha_{(k_1,k_2)} = \alpha_{(1,1)} \) (respectively \( \beta_{(k_1,k_2)} = \beta_{(1,1)} \)) for all \( k_1, k_2 \geq 1 \). We say that \( T \) is flat if \( T \) is horizontally and vertically flat, and we say that \( T \) is symmetrically flat if \( T \) is flat and \( \alpha_{(1,1)} = \beta_{(1,1)} \).
We consider the sequences \( \{X_n\}_{n \geq 0} \) and \( \{Y_{(i,n)}\}_{n \geq 0} \) \((i \geq 0 \text{ fixed})\) defined by setting

\[
X_n := \begin{cases} 
1 & \text{if } n = 0 \\
\alpha_n^2 & \text{if } n > 0 
\end{cases} \quad Y_{(i,n)} := \begin{cases} 
1 & \text{if } n = 0 \\
\beta_{i,n}^2 & \text{if } n > 0 
\end{cases} \tag{5}
\]

It’s obvious to see that the 2-variable weighted shift, \( Y \), is flat. Moreover, we have

\[
X_n Y_{(i,n)} = \sum_{j=0}^{i-1} \rho_{(i,j)} \lambda_j^n
\]

that, for a subnormal weighted shift \( W \), the 2-variable weighted shift, \( X \), we have the following result, using the Binet formula. In the case of 2-variable weighted shift, every \( k \in \mathbb{Z}^+ \), the \( \gamma_k \) is expressed in terms of \( X_n \) and \( Y_{(i,n)} \):

\[
\gamma_k \equiv \gamma_k(\alpha, \beta) := \begin{cases} 
1 & \text{if } k = (0,0); \\
X_{k_1} & \text{if } k_1 \geq 1 \text{ and } k_2 = 0; \\
Y_{(0,k_2)} & \text{if } k_1 = 0 \text{ and } k_2 \geq 1; \\
X_{k_1} Y_{(k_1,k_2)}, & \text{for } k_1 \geq 1, k_2 \geq 1.
\end{cases} \tag{6}
\]

In addition, we suppose that \( X_n = \sum_{i=0}^{s-1} a_i t_i^n \) \((a_i \in \mathbb{R}, t_i \in \mathbb{R}^+)\) and \( Y_{(i,n)} = \sum_{j=0}^{r-1} \rho_{(i,j)} \lambda_j^n \) where \( \rho_{(i,j)} = \rho_{(0,j)} \lambda_j^i \) with \( \rho_{(i,j)} \in \mathbb{R} \) and \( \lambda_j \in \mathbb{R}^+ \).

As a first step, since \( \alpha_{(k_1,0)} \) is given, we build the sequence \( \{\alpha_{(k_1,k_2)}\}_{k_2 \in \mathbb{N}} \) for every \( k_1 \geq 0 \), by employing the commutativity condition (3). That is, setting \( X_{(k_1,k_2)} = \alpha_{(0,k_1)}^2 \cdot \alpha_{(1,k_2)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,k_2)}^2 \), it turns out that, due to the commutativity condition (3), an induction process permits us to obtain \( \alpha_{(k_1,k_2)} = \frac{\beta_{(k_1,k_2+1)}^2}{\beta_{(k_1,k_2)}^2} \). Thus we can show that \( X_{(k_1,k_2)} = \frac{X_{k_1} Y_{(k_1,k_2)}}{Y_{(0,k_2)}} \) and it follows that

\[
\alpha_{(k_1,k_2)}^2 = \alpha_{(k_1,0)}^2 \cdot \frac{Y_{(k_1+1,k_2)}}{Y_{(k_1,k_2)}}. \tag{7}
\]

Expression (7) is very useful for studying flatness of weight sequences (6) for the 2-variable weighted shift \( T = (T_1, T_2) \). Moreover, Stampfli showed in [14] that, for a subnormal weighted shift \( W_\alpha \), a propagation phenomenon occurs which forces the flatness of \( W_\alpha \) whenever two equals weight are present. For the truncated case, it was shown in [4] that recursiveness permits to establish this result, using the Binet formula. In the case of 2-variable weighted shift, we have the following result,

**Theorem 2.3** Let \( T = (T_1, T_2) \) be a 2-variable weighted shift whose weight sequence \( \gamma \equiv \{\gamma_k\}_{k \in \mathbb{Z}^+} \) is given by (6). Suppose that \( \gamma_k \) is expressed in terms of \( X_n \) and \( Y_{(i,n)} \) such that \( X_n = \sum_{i=0}^{s-1} a_i t_i^n \), with \( a_i \in \mathbb{R} \) and \( t_i \in \mathbb{R}^+ \) are distinct, and \( Y_{(i,n)} = \sum_{j=0}^{r-1} \rho_{(i,j)} \lambda_j^n \) where \( \rho_{(i,j)} = \rho_{(0,j)} \lambda_j^i \), with \( \rho_{(i,j)} \in \mathbb{R} \) and \( \lambda_j \in \mathbb{R}^+ \) are distinct. If \( \beta_{(0,n)} = \beta_{(0,n+1)} \), for some \( n \geq 1 \), then \( T = (T_1, T_2) \) is vertically flat. Moreover, we have \( \beta_{(k_1,k_2)} = \beta_{(0,1)} \) and \( \alpha_{k+e_2} = \alpha_k \).
Proof. Adopting the process of the demonstration of Proposition 4.5 of [4], the equality \( \beta(0,n) = \beta(0,n+1) \) leads to have \( \lambda_j = \lambda_h = \lambda \) for all \( j, h \) \( (0 \leq j, h \leq r-1) \). We thus obtain that \( Y_{(0,n)} = \sum_{j=0}^{r-1} \rho_{(0,j)} \lambda^n = \rho \lambda^n \).

Since \( Y_{(i,n)} = \sum_{j=0}^{r-1} \rho_{(i,j)} \lambda^n \), it follows that \( Y_{(i,n)} = \rho \lambda^{n+i} \), which implies that \( \beta_{(k_1,k_2)} = \sqrt{\lambda} \) for \( k_1, k_2 \geq 0 \) with \( (k_1,k_2) \neq (0,0) \). This can be expressed as saying that \( \beta_{k+\epsilon} = \beta_k \) for \( k \in \mathbb{Z}^2_+ \), \( k \neq (0,0) \) and \( \epsilon = (1,0), (0,1) \). Previously, by the commutativity condition (3), we had established that \( \alpha_{(k_1,k_2)} \) is given by (7), therefore it follows that \( \alpha_{(k_1,k_2)}^2 = \alpha_{(k_1,0)}^2 \lambda_{k_1} \), and we conclude that \( \alpha_{k+\epsilon} = \alpha_k. \square \)

We now infer the following basic proposition, on the closed relationship between the class of subnormal 2-variable weighted shift released from the linear recursive relation (2).

**Theorem 2.4** Let \( T = (T_1, T_2) \) be a 2-variable weighted shift with weight sequences \( \gamma \equiv \{ \gamma_k \}_{k \in \mathbb{Z}^2_+} \) given by (6). We assume that \( \gamma_k \) is expressed in terms of \( X_n \) and \( Y_{(i,n)} \) such that \( X_n = \sum_{i=0}^{s-1} a_i t_i^n \) with \( a_i \in \mathbb{R} \) and \( t_i \in \mathbb{R}^+ \) are distinct, and \( Y_{(i,n)} = \sum_{j=0}^{r-1} \rho_{(i,j)} \alpha_j^n \), where \( \rho_{(i,j)} = \rho_{(0,j)} \alpha_j^i \) with \( \rho_{(0,j)} \in \mathbb{R} \) and \( \lambda_j \in \mathbb{R}^+ \) are distinct. Then \( T \) is subnormal, if and only if, \( a_i \) \( (0 \leq i \leq s-1) \) and \( \rho_{(0,j)} \) \( (0 \leq j \leq r-1) \) are positive.

Proof. For \( i \geq 0 \) fixed, we have \( Y_{(i,n)} = \sum_{j=0}^{r-1} \rho_{(i,j)} \alpha_j^n \). The Binet formula allows us to deduce that \( \{ Y_{(i,n)} \}_{n \geq 0} \) is nothing else but only the Fibonacci sequence (2), whose associated characteristic polynomial is \( P(z) = \prod_{i=0}^{r-1} (z - \lambda_i) \). From Theorem 2.3 it turns out that the sequence \( \{ Y_{(i,n)} \}_{n \geq 0} \) admits an \( r \) atoms representing measure \( \mu_i = \sum_{j=0}^{r-1} \rho_{(i,j)} \delta_{\lambda_j} \), since \( \rho_{(i,j)} \in \mathbb{R}^+ \). As well as, the sequence \( \{ X_n \}_{n \geq 0} \) is a Fibonacci sequence (2) whose associated characteristic polynomial is \( P(z) = \prod_{i=0}^{r-1} (z - t_i) \) and the coefficients \( a_i \) \( (0 \leq i \leq s-1) \) are positive. Therefore, the sequence \( \{ X_n \}_{n \geq 0} \) admits an \( s \) atoms representing measure \( \nu_i = \sum_{j=0}^{r-1} a_i \delta_{t_i} \). As a consequence, we obtain \( \gamma_k = \gamma_{(k_1,k_2)} \equiv \gamma_k(\alpha, \beta) = \sum_{i=0}^{s-1} \rho_{(i,j)} a_i \delta_{t_i} \alpha_j^{k_1+k_2} \). Thereby, we have

\[
\gamma_{(k_1,k_2)} = \int \int_{\mathbb{R}^2} x^{k_1} y^{k_2} d\mu(x,y),
\]

where \( \mu(x,y) = \sum_{i=0}^{s-1} \sum_{j=0}^{r-1} \rho_{(i,j)} a_i \delta_{t_i} \alpha_j^{k_1+k_2} (x) \delta_{\lambda_j} (y) \) and \( Supp(\mu) \in [0, Sup(t_i \lambda_j)] \times [0, Sup(\lambda_j)] \) \( (0 \leq i \leq s-1 \) and \( 0 \leq j \leq r-1) \). \( \square \)

Remind that the study of subnormality for multivariable weighted shifts is intimately connected to multivariable real moment problems, therefore in light of Theorem 2.4, we arrive to obtain the connection between the 2-variable weighted shift with weight sequence (6) and the one variable moment problem.

**Theorem 2.5** Let \( T = (T_1, T_2) \) be a 2-variable weighted shifts with weight sequences \( \gamma \equiv \{ \gamma_k \}_{k \in \mathbb{Z}^2_+} \) given by (6). We assume that \( \gamma_k \) is expressed in...
terms of \( X_n \) and \( Y_{(i,n)} \) such that \( X_n = \sum_{i=0}^{s-1} a_i t_i^n \) (with \( a_i \in \mathbb{R} \) and \( t_i \in \mathbb{R}^+ \) are distinct) and \( Y_{(i,n)} = \sum_{j=0}^{r-1} \rho_{(i,j)} \lambda_j^n \) where \( \rho_{(i,j)} = \rho_{(0,j)} \lambda_j^i \) (with \( \rho_{(i,j)} \in \mathbb{R} \) and \( \lambda_j \in \mathbb{R}^+ \) are distinct). Then \( T \) is subnormal, if and only if, \( \{X_n\}_{n \geq 0} \) and \( \{Y_{(0,n)}\}_{n \geq 0} \) are moment sequences.

Results of Theorems 2.4 and 2.5 illustrate our motivation to study the characterization of the subnormality for 2-variable weighted shifts, in terms of recursive relation of Fibonacci type. In the next section, we pause to paraphrase the contribution of our approach to interpret the subnormality via the positivity of Hankel matrices.

3 The subnormal 2-variable weighted shifts and the Hankel matrices

Recall that the notion of \( k \)-hyponormality, which is intimately related to the positivity of the matrix \( M_u(k) := (Y_{u+(m,n)+(p,q)})_{0 \leq m+n \leq k, 0 \leq p+q \leq k} \) for all \( u \in \mathbb{Z}_+^2 \), is fundamental in the Curto-Fialkow’s theory for insuring the characterization of subnormality for 2-variable weighted shifts. By considering the notion of Hankel matrix \( H(s) = (\omega_{i+j})_{0 \leq i,j \leq s} \) associated with an usual sequence \( \{\omega_n\}_{n \geq 0} \), we get the following result.

Theorem 3.1 Let \( T = (T_1, T_2) \) be a 2-variable weighted shifts with weight sequences \( \{\gamma_k\}_{k \in \mathbb{Z}_+^2} \) given by (6). We assume that \( \gamma_k \) is expressed in terms of \( X_n \) and \( Y_{(i,n)} \) such that \( X_n = \sum_{i=0}^{s-1} a_i t_i^n \) (with \( a_i \in \mathbb{R} \) and \( t_i \in \mathbb{R}^+ \) are distinct) and \( Y_{(i,n)} = \sum_{j=0}^{r-1} \rho_{(i,j)} \lambda_j^n \) where \( \rho_{(i,j)} = \rho_{(0,j)} \lambda_j^i \) (with \( \rho_{(i,j)} \in \mathbb{R} \) and \( \lambda_j \in \mathbb{R}^+ \) are distinct). The following statements are equivalent,

(i) \( T \) is subnormal;
(ii) \( H_1(k) = (X_{i+j})_{0 \leq i,j \leq k} \) and \( H_2(m) = (Y_{(0,i+j)})_{0 \leq i,j \leq m} \) are positive for any \( k,m \geq 0 \).

Proof. Later, Curto-Fialkow showed in [7], [8] that if \( H_1(k) = (X_{i+j})_{0 \leq i,j \leq k} \geq 0 \) for every \( k \geq 0 \), then \( \{X_n\}_{n \geq 0} \) is a moment sequence, likewise \( H_2(m) = (Y_{(0,i+j)})_{0 \leq i,j \leq m} \geq 0 \) for every \( m \geq 0 \), thus \( \{Y_{0,n}\}_{n \geq 0} \) is also a moment sequence. As a matter of fact, \( a_i \) \((0 \leq i \leq s-1)\) and \( \rho_{(0,j)} \) \((0 \leq j \leq r-1)\) are positive, and in light of Theorem 2.4 we arrive to establish that \( T \) is subnormal. \( \Box \)

By combining this latter result with Proposition 3.2 in [4], where we utilize the Hankel matrix’s techniques, we arrive to weak the hypothesis of the precedent theorem and we get the following result.

Theorem 3.2 Let \( T = (T_1, T_2) \) be a 2-variable weighted shifts with weight sequence \( \{\gamma_k\}_{k \in \mathbb{Z}_+^2} \) given by (6). We assume that \( \gamma_k \) is expressed in terms of \( X_n \) and \( Y_{(i,n)} \) such that \( X_n = \sum_{i=0}^{s-1} a_i t_i^n \) (with \( a_i \in \mathbb{R} \) and \( t_i \in \mathbb{R}^+ \) are distinct)
and \( Y_{(i,n)} = \sum_{j=0}^{r-1} \rho_{(i,j)} \lambda_j^i \) where \( \rho_{(i,j)} = \rho_{(0,j)} \lambda_j^i \) (with \( \rho_{(i,j)} \in \mathbb{R} \) and \( \lambda_j \in \mathbb{R}^+ \) are distinct). The following statements are equivalent:

(i) \( T \) is subnormal;
(ii) \( H_1(s-1) \geq 0 \) and \( H_2(r-1) \geq 0 \).

By applying the latter Theorem 3.2, we infer the characterization of the \( k \)-hyponormality for \( 2 \)-variable weighted shifts with weight sequence \( \{ \gamma_k \} \), given by (6), in terms of the positivity of \( H_1(s-1) \geq 0 \) and \( H_2(r-1) \geq 0 \) and we get the result,

**Theorem 3.3** Let \( T = (T_1, T_2) \) be a \( 2 \)-variable weighted shifts with weight sequence \( \gamma \equiv \{ \gamma_k \}_{k \in \mathbb{Z}_+^2} \) given by (6). We assume that \( \gamma_k \) is expressed in terms of \( X_n \) and \( Y_{(i,n)} \) such that \( X_n = \sum_{i=0}^{n-1} a_i t_i^n \) (with \( a_i \in \mathbb{R} \) and \( t_i \in \mathbb{R}^+ \) are distinct) and \( Y_{(i,n)} = \sum_{j=0}^{r-1} \rho_{(i,j)} \lambda_j^i \) where \( \rho_{(i,j)} = \rho_{(0,j)} \lambda_j^i \) (with \( \rho_{(i,j)} \in \mathbb{R} \) and \( \lambda_j \in \mathbb{R}^+ \) are distinct). The following statements are equivalent:

(i) \( T \) is subnormal;
(ii) \( T \) is \( k \)-hyponormal for all \( k \in \mathbb{Z}_+^2 \);
(iii) \( (Y_u Y_{u+(m,n)+(p,q)} - Y_{u+(m,n)} Y_{u+(p,q)}))_{1 \leq m+n \leq k, 1 \leq p+q \leq k} \geq 0 \) for all \( k \in \mathbb{Z}_+^2 \) and \( u \in \mathbb{Z}_+^2 \);
(iv) \( M_u(k) := (Y_u Y_{u+(m,n)+(p,q)})_{0 \leq m+n \leq k, 0 \leq p+q \leq k} \geq 0 \) for all \( k \in \mathbb{Z}_+^2 \) and \( u \in \mathbb{Z}_+^2 \);
(v) \( H_1(s-1) \geq 0 \) and \( H_2(r-1) \geq 0 \).

**Remark 3.4** Two results characterizing the \( k \)-hyponormality and the criterion for subnormality of \( 2 \)-variable weighted shifts have been formulated by Curto et al.. The first one, concerns the \( k \)-hyponormality and the positivity of some matrices and the second one concerns the subnormality (see [9]).

In [4] an important class of subnormal weighted shifts is explored by considering measures \( \mu \) with two atoms \( \lambda_1 \) and \( \lambda_2 \). A sequence \( \{ \gamma_n \}_{n \geq 0} \) such that \( \gamma_0 = 1, \gamma_1 \) (given) and \( \gamma_{n+1} = a_0 \gamma_n + a_1 \gamma_{n-1} \) (for \( n \geq 1 \)) is a moment sequence, if and only if, \( P(\gamma_1) \leq 0 \). As a matter of fact, we conclude this section by studying the fallout of our approach on this class of subnormal weighted shifts. Let \( T = (T_1, T_2) \) be a \( 2 \)-variable weighted shifts with weight sequences \( \{ \gamma_k \} \), given by (6), expressed in terms of \( \{ X_n \}_{n \geq 0} \) and \( \{ Y_{(i,n)} \}_{n \geq 0} \) (\( i \geq 0 \) fixed) such that \( \{ X_n \}_{n \geq 0} \) and \( \{ Y_{(0,n)} \}_{n \geq 0} \) are two Fibonacci sequences (2) of order 2.

Involving the study provided in [4], when combined with the aforementioned result of Theorem 3.3, we obtain the following application.

**Theorem 3.5** Let \( T = (T_1, T_2) \) be a \( 2 \)-variable weighted shifts with weight sequence \( \gamma \equiv \{ \gamma_k \}_{k \in \mathbb{Z}_+^2} \) given by (6). We assume that \( \gamma_k \) is expressed in terms of \( X_n \) and \( Y_{(i,n)} \) satisfying \( X_n = b_1 X_{n-1} + b_2 X_{n-2} \), \( Y_{(0,n)} = c_1 Y_{(0,n-1)} + c_2 Y_{(0,n-2)} \) and \( Y_{(i,n)} = \sum_{j=0}^{r-1} \rho_{(i,j)} \lambda_j^{n} \) where \( \rho_{(i,j)} = \rho_{(0,j)} \lambda_j^i \) with \( \rho_{(i,j)} \in \mathbb{R} \). Suppose that the characteristic roots \( t_1, t_2 \) and \( \lambda_1, \lambda_2 \) of \( P_1(z) = z^2 - b_1 z - b_2 \) \( P_2(z) = z^2 - c_1 z - c_2 \).
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(respectively) are positive. Then, the following statements are equivalent.
(i) $T$ is subnormal;
(ii) $T$ is $k$-hyponormal for all $k \in \mathbb{Z}_+$;
(iii) $H_1(1) \geq 0$ and $H_2(1) \geq 0$;
(iv) $P_1(\alpha_{(0,0)}^2) \leq 0$ and $P_2(\beta_{(0,0)}^2) \leq 0$;
(v) $\alpha_{(0,0)}^2 \in [t_1, t_2]$ and $\beta_{(0,0)}^2 \in [\lambda_1, \lambda_2]$.

We show that Theorem 3.5 provides a concrete criterion of the subnormality for the 2-variable weighted shift operator $T = (T_1, T_2)$, in terms of the characteristic polynomials $P_1$ and $P_2$ of the sequences $\{X_n\}_{n \geq 0}$ and $\{Y_{(i,n)}\}_{n \geq 0}$ defined by (5).

References


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