On Stability of Three Interacting Species

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Abstract

Using Liapunov’s direct method, the stability of the equilibrium state of a system of three interacting species living in a finite one-dimensional linear habitat has been investigated. The effect of convective and dispersive migration on the stability of the system has also been studied. It has been shown that, for similar dispersion coefficients, the equilibrium state remains stable without dispersal as well as with dispersal. The stability or instability of the equilibrium state of the system is not affected by convective migration. The dispersal has stabilizing effect even on the nonlinear stability of the equilibrium state of the interacting system.

Keywords: Liapunov’s method, dispersal, migration, global stability, stable equilibrium, prey, predator

1 Introduction

The relationship between predators and their prey has been investigated widely over years, and will continue to be one of the most important topics in mathematical ecology and theoretical biology due to its universal existence as well as its importance in studying the equilibrium, stability, and persistence of ecological system. Fan and Li \cite{FanLi} and Wang and Fan \cite{WangFan} investigated the permanence and stability in delayed ratio-dependent predator-prey systems. Existence of positive periodic solutions for the delayed ratio-dependent predator-prey system was presented by Fan, Li, and Wang \cite{FanLiWang}, Fan, Wang, and Zou \cite{FanWangZou}, and Li and Wang \cite{LiWang}.

Stability of systems of interacting species has been investigated by various authors, for example, Goh and Agnew \cite{GohAgnew}; Kazarinoff and Driesshe \cite{KazarinoffDriesshe}; El-Owaidy and Ammar \cite{ElOwaidyAmmar}.

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Effects of dispersal on the stability of the equilibrium state have also been investigated by various authors, for example, Chow and Tam [2], Comina and Blatt [3], Gopalsamy [7], Hastings [8], Kramer [10], Levin [11], Nallaswamy [13], Nallaswamy and Shukla [14], El-Owaidy and Ammar [5], Ammar [1].

Effects of convective migration on the species evolution have been considered by various investigators "for example": Comina and Blatt [3], Gopalsamy [7], Memurtrie [12], Nallaswamy [13], Nallaswamy and Shukla [15], Shukla and Das [17], M. Kot [18].

In particular, Nallaswamy [16] investigated the effect of convective and dispersive migration on the global stability of the equilibrium state of system of two interacting species, and Ammar [1] investigated the effect of dispersal on the linear and non-linear stability of the equilibrium state of two predators and a common prey system with response.

Keeping the above into consideration, we discuss in this paper the effect of convective and dispersive migration on the stability of a system of three interacting species living in a finite one dimensional linear habitat. It is shown that stability or instability of the equilibrium state of the system is not affected by convective migration of the species. The dispersal has stabilizing effect even on the nonlinear stability of the equilibrium state of the interacting system.

2 Mathematical Model

Consider the interactions of three species living in a finite \(0 \leq x \leq L\) one dimensional linear habitat. The system governing the evolution of the species with convective and dispersive migration can be written (Nallaswamy [16]) as:

\[
\begin{align*}
\frac{\partial N_1}{\partial t} + \frac{\partial N_1}{\partial x} &= N_1 f_1(N_1, N_2, N_3) + \frac{\partial}{\partial x}(D_1 \frac{\partial N_1}{\partial x}), \\
\frac{\partial N_2}{\partial t} + \frac{\partial N_2}{\partial x} &= N_2 f_2(N_1, N_2, N_3) + \frac{\partial}{\partial x}(D_2 \frac{\partial N_2}{\partial x}), \\
\frac{\partial N_3}{\partial t} + \frac{\partial N_3}{\partial x} &= N_3 f_3(N_1, N_2, N_3) + \frac{\partial}{\partial x}(D_3 \frac{\partial N_3}{\partial x}),
\end{align*}
\]

(2.1)

where \(N_1(x, t), N_2(x, t), N_3(x, t)\) represent the population distributions of three species at time \(t\), and the functions \(f_1, f_2, f_3\) represent the per capita growth rate of the species, \(D_1, D_2, D_3\) represent the positive variable dispersal coefficients of species, and \(v_1, v_2, v_3\) are their convective velocities.

The nontrivial positive equilibrium state \((N_1^*, N_2^*, N_3^*)\) of system (2.1) is given by

\[
\begin{align*}
f_1(N_1^*, N_2^*, N_3^*) &= 0, \\
f_2(N_1^*, N_2^*, N_3^*) &= 0, \\
f_3(N_1^*, N_2^*, N_3^*) &= 0.
\end{align*}
\]

(2.2)
Without loss of generality, assume that there exist only one nontrivial positive equilibrium state. Writing
\[ N_i(x, t) = N'_i + n_i(x, t), \quad i = 1, 2, 3 \] (2.3)
in system (2.1) and keeping (2.2) in view, we get the following nonlinear equations:
\[
\begin{align*}
\frac{\partial n_1}{\partial t} + v_1 \frac{\partial n_1}{\partial x} &= (N'_1 + n_1) g_1(n_1, n_2, n_3) + \frac{\partial}{\partial x} \left( D_1 \frac{\partial n_1}{\partial x} \right) \\
\frac{\partial n_2}{\partial t} + v_2 \frac{\partial n_2}{\partial x} &= (N'_2 + n_2) g_2(n_1, n_2, n_3) + \frac{\partial}{\partial x} \left( D_2 \frac{\partial n_2}{\partial x} \right) \\
\frac{\partial n_3}{\partial t} + v_3 \frac{\partial n_3}{\partial x} &= (N'_3 + n_3) g_3(n_1, n_2, n_3) + \frac{\partial}{\partial x} \left( D_3 \frac{\partial n_3}{\partial x} \right)
\end{align*}
\] (2.4)
where
\[ g_i(n_1, n_2, n_3) = f_i(N'_1 + n_1, N'_2 + n_2, N'_3 + n_3), \quad i = 1, 2, 3 \] (2.5)
Notice that the equilibrium state (2.3) implies that \( N_i(0, t) = N_i(L, t) = N_i(0, 0) \) since the solution \( N'_i \) exists. The system (2.4) is associated with the following boundary conditions:
\[ (i) \text{ Flux Conditions:} \]
\[ \frac{\partial n_i}{\partial x}(0, t) = \frac{\partial n_i}{\partial x}(L, t), \quad i = 1, 2, 3 \] (2.6)
These conditions imply that there is no migration of the species across the boundary of habitat. That is, these conditions suggest that the species populations remain at equilibrium level on the boundary of the habitat.
\[ (ii) \text{ Reservoir Conditions:} \]
\[ n_i(0, t) = n_i(L, t) = 0, \quad i = 1, 2, 3 \] (2.7)
These conditions represent the fact that the species population at the boundary are equal to their equilibrium values.
In the following, the global stability of the equilibrium state \( (N'_1, N'_2, N'_3) \) of system (2.1) is analyzed using Liapunov’s direct method.

3 Stability
To investigate the global stability of the equilibrium state in the absence of convective and dispersal migration, consider the positive definite function:
\[ E(n_1, n_2, n_3) = n_1 - N_1^* \ln \left(1 + \frac{n_1}{N_1^*}\right) + c_2(n_2 - N_2^* \ln \left(1 + \frac{n_2}{N_2^*}\right)) + c_3(n_3 - N_3^* \ln \left(1 + \frac{n_3}{N_3^*}\right)) \]

where \( c_2 \) and \( c_3 \) are positive constants to be chosen appropriately. Then using (2.4),

\[ \frac{dE}{dt} = n_1 g_1(n_1, n_2, n_3) + c_2 n_2 g_2(n_1, n_2, n_3) + c_3 n_3 g_3(n_1, n_2, n_3). \]  

By choosing \( c_2 \) and \( c_3 \) appropriately, it is noted from (3.2) that the equilibrium state of the system (2.1) is globally asymptotically stable in the positive quadrant-octant

\[ p = \{(N_1, N_2, N_3); N_i > 0, i = 1, 2, 3\} \]

of the phase plane provided \( \frac{dE}{dt} < 0 \) for \( n_1 \neq 0, n_2 \neq 0, n_3 \neq 0 \) and \( \frac{dE}{dt} = 0 \) only when \( n_1 = n_2 = n_3 = 0 \). If these conditions are satisfied in any subregion \( A \) (containing the equilibrium state) of the positive quadrant then the equilibrium state is nonlinearly asymptotically stable in the region \( A \). Effects of convective and dispersive migration of the global stability of the equilibrium state are studied by considering the positive definite function

\[ E(n_1, n_2, n_3) = \int_0^L \{n_1 - N_1^* \ln \left(1 + \frac{n_1}{N_1^*}\right) + c_2(n_2 - N_2^* \ln \left(1 + \frac{n_2}{N_2^*}\right)) + c_3(n_3 - N_3^* \ln \left(1 + \frac{n_3}{N_3^*}\right)\} ds \]

where \( L \) is the length of the habitat and the constants \( c_2, c_3 \) are the same as chosen in (3.1). Then using (2.4)

\[ \frac{dE}{dt} = \int_0^L \left[n_1 g_1(n_1, n_2, n_3) + c_2 n_2 g_2(n_1, n_2, n_3) + c_3 n_3 g_3(n_1, n_2, n_3)\right] dx \]

\[ -v_1 \int_0^L \left[\frac{n_1}{N_1^* + n_1} \frac{\partial n_1}{\partial x} - \frac{n_2}{N_2^* + n_2} \frac{\partial n_2}{\partial x}\right] dx \]

\[ -v_3 \int_0^L \left[\frac{n_3}{N_3^* + n_3} \frac{\partial n_3}{\partial x}\right] dx + \int_0^L \left[\frac{n_1}{N_1^* + n_1} \frac{\partial}{\partial x} \left(D_1 \frac{\partial n_1}{\partial x}\right)\right] dx \]

\[ + \int_0^L \left[\frac{n_2}{N_2^* + n_2} \frac{\partial}{\partial x} \left(D_2 \frac{\partial n_2}{\partial x}\right)\right] dx + \int_0^L \left[\frac{n_3}{N_3^* + n_3} \frac{\partial}{\partial x} \left(D_3 \frac{\partial n_3}{\partial x}\right)\right] dx. \]
Evaluating the last six integrals in (3.4) and making use of homogeneous 
boundary conditions (2.6), we have

\[
\frac{dE}{dt} = \int_0^L \left[ n_1 g_1(n_1, n_2, n_3) + c_2 n_2 g_2(n_1, n_2, n_3) + c_3 n_3 g_3(n_1, n_2, n_3) \right] dx \\
- \int_0^L \frac{D_1 N_1^*}{(N_1^* + n_1)^2} (\frac{\partial n_1}{\partial x})^2 dx - c_2 \int_0^L \frac{D_2 N_2^*}{(N_2^* + n_2)^2} (\frac{\partial n_2}{\partial x})^2 dx \\
- c_3 \int_0^L \frac{D_3 N_3^*}{(N_3^* + n_3)^2} (\frac{\partial n_3}{\partial x})^2 dx.
\]

If the first, second and third integrals in (3.5) are nonpositive, it follow
that \( \frac{dE}{dt} < 0 \) for \( n_1, n_2, n_3 > 0 \) and \( \frac{dE}{dt} = 0 \) only when \( n_1 = n_2 = n_3 = 0 \).
Hence the stable equilibrium state without dispersal remains so with
dispersal. Since the terms which arise due to convective migration vanish,
it is also concluded that stability or instability of the equilibrium state
of the system is not affected by convective migration.
The above results hold for any subregion of the feasible region, i.e., if
the equilibrium state is nonlinearly asymptotically stable in the absence
of migration in any subregion of the feasible region, then it continues to
be stable with migration also.

4 Example

Consider the interaction of two preys \((N_1(x, t), N_2(x, t))\) and one predator
\((N_3(x, t))\).
Then the system is given by

\[
\frac{\partial N_1}{\partial t} = N_1(a_1 - a_{11} N_1 - \frac{a_{13} N_3}{1 + \alpha N_1}) + \frac{\partial}{\partial x} \left( D_1 \frac{\partial N_1}{\partial x} \right) \\
\frac{\partial N_2}{\partial t} = N_2(a_2 - a_{22} N_2 - \frac{a_{23} N_3}{1 + \alpha N_2}) + \frac{\partial}{\partial x} \left( D_2 \frac{\partial N_2}{\partial x} \right) \\
\frac{\partial N_3}{\partial t} = N_3(-a_3 + \frac{a_{31} N_1}{1 + \alpha N_1} + \frac{a_{32} N_2}{1 + \alpha N_2} - a_{33} N_3) + \frac{\partial}{\partial x} \left( D_3 \frac{\partial N_3}{\partial x} \right)
\]

(4.1)

where the interaction coefficient \( a_1, a_2, a_3, a_{11}, a_{22}, a_{33}, a_{13}, a_{23}, a_{31}, a_{32} \) and
\( \alpha \) are positive constants. The factor \((1 + \alpha N_1)^{-1}\) and \((1 + \alpha N_2)^{-1}\) repre-
sent the functional response in the model and \( \alpha \) determines the strength
of this response.
The nontrivial positive equilibrium state \((N_1^*, N_2^*, N_3^*)\) is obtained from

\[
\begin{align*}
(a_1 - a_{11}N_1^*)(1 + aN_1^*) &= a_{13}N_3^* \\
(a_2 - a_{22}N_2^*)(1 + aN_2^*) &= a_{23}N_3^* \\
(-a_3 - a_{33}N_3^*)(1 + aN_1^*)(1 + aN_2^*) &= a_{31}N_1^*(1 + aN_2^*) + a_{32}N_2^*(1 + aN_1^*)
\end{align*}
\]  

(4.2)

using (2.3) in (4.1) we have

\[
\begin{align*}
\frac{\partial n_1}{\partial t} &= \frac{N_1^* + n_1}{s_1} \left\{ (\alpha a_{13}N_3^* - a_{11}s_1)n_1 - a_{13}(1 + \alpha N_1^*)n_3 \right\} + \frac{\partial}{\partial x} \left( D_1 \frac{\partial n_1}{\partial x} \right) \\
\frac{\partial n_2}{\partial t} &= \frac{N_2^* + n_2}{s_2} \left\{ (\alpha a_{23}N_3^* - a_{22}s_2)n_2 - a_{23}(1 + \alpha N_2^*)n_3 \right\} + \frac{\partial}{\partial x} \left( D_2 \frac{\partial n_2}{\partial x} \right) \\
\frac{\partial n_3}{\partial t} &= \frac{N_3^* + n_3}{s_1s_2} \left\{ a_{31}s_2n_1 + a_{32}s_1n_2 - a_{33}s_1s_2n_3 \right\} + \frac{\partial}{\partial x} \left( D_3 \frac{\partial n_3}{\partial x} \right)
\end{align*}
\]

(4.3)

where

\[
\begin{align*}
s_1 &= (1 + \alpha N_1^*)(1 + \alpha N_1^* + \alpha n_1) \\
s_2 &= (1 + \alpha N_2^*)(1 + \alpha N_2^* + \alpha n_2)
\end{align*}
\]

The linearized version of system (4.3) is

\[
\begin{align*}
\frac{\partial n_1}{\partial t} &= -k_{11}n_1 - k_{13}n_3 + \frac{\partial}{\partial x} \left( D_1 \frac{\partial n_1}{\partial x} \right) \\
\frac{\partial n_2}{\partial t} &= -k_{22}n_2 - k_{23}n_3 + \frac{\partial}{\partial x} \left( D_2 \frac{\partial n_2}{\partial x} \right) \\
\frac{\partial n_3}{\partial t} &= k_{31}n_1 + k_{32}n_2 - k_{33}n_3 + \frac{\partial}{\partial x} \left( D_3 \frac{\partial n_3}{\partial x} \right)
\end{align*}
\]

(4.4)

where

\[
\begin{align*}
k_{11} &= \frac{2\alpha a_{11}N_1^*}{1 + \alpha N_1^*}(N_1^* - a), \quad a = \frac{\alpha a_{11} - a_{11}}{2\alpha a_{11}}
\end{align*}
\]

we shall assume that \(a > 0\), it’s clear that \(k_{11} \geq 0\) for \(N_1^* \geq a\) and \(k_{11} < 0\) for \(0 < N_1^* < a\)

\[
k_{13} = \frac{a_{13}N_1^*}{1 + \alpha N_1^*} > 0, \quad k_{22} = \frac{2\alpha a_{22}N_2^*}{1 + \alpha N_2^*}(N_2^* - b), \quad b = \frac{\alpha a_{22} - a_{22}}{2\alpha a_{22}}
\]

we shall assume that \(b > 0\) and \(k_{22} \geq 0\) for \(N_2^* \geq b\) and \(k_{22} < 0\) for \(0 < N_2^* < b\).

\[
\begin{align*}
k_{23} &= \frac{a_{23}N_2^*}{1 + \alpha N_2^*} > 0 \\
k_{31} &= \frac{a_{31}N_3^*}{(1 + \alpha N_1^*)^2} > 0 \\
k_{32} &= \frac{a_{32}N_3^*}{(1 + \alpha N_2^*)^2} > 0 \\
k_{33} &= a_{33}N_3^* > 0
\end{align*}
\]

(4.5)
To investigate the linear and non-linear stability of the equilibrium state, we have the following theorems:

**Theorem 4.1**

(a) If \( N_1^* \geq a \) and \( N_2^* \geq b \), the equilibrium point \( E(N_1^*, N_2^*, N_3^*) \) of system (4.1) is linear asymptotically stable in the entire positive octant (with and without dispersal).

(b) If \( N_1^* < a \) and \( N_2^* < b \), the equilibrium point \( E(N_1^*, N_2^*, N_3^*) \) of system (4.1) is not linear asymptotically stable without dispersal, but with constant dispersal, it does become stable under the two conditions:

\[
D_1 \geq \frac{2\alpha a_{11} L^2 N_1^* (a - N_1^*)}{\Pi^2 (1 + \alpha N_1^*)} \quad \text{and} \quad \frac{2\alpha a_{22} L^2 N_2^* (b - N_2^*)}{\Pi^2 (1 + \alpha N_2^*)}
\]

(c) If \( N_1^* < a \) and \( N_2^* \geq b \) \((N_1^* \geq a \) and \( N_2^* < b \)) the equilibrium point \( E(N_1^*, N_2^*, N_3^*) \) of system (4.1) is not linear asymptotically stable without dispersal, but with constant dispersal, it does become stable under the two conditions:

\[
D_1 \geq \frac{2\alpha a_{11} L^2 N_1^* (a - N_1^*)}{\Pi^2 (1 + \alpha N_1^*)} \quad \text{and} \quad D_2 \geq \frac{2\alpha a_{22} L^2 N_2^* (b - N_2^*)}{\Pi^2 (1 + \alpha N_2^*)}
\]

**Proof**

(a) To study the stability of the equilibrium state in the absence of dispersal, consider the positive definite function:

\[
E = \frac{1}{2} (c_1 n_1^2 + c_2 n_2^2 + n_3^2) \quad (4.6)
\]

where \( c_1 \) and \( c_2 \) are positive constants to be chosen appropriately.

Then using (4.4) and choosing \( c_1 = \frac{k_{31}}{k_{13}} \) and \( c_2 = \frac{k_{32}}{k_{23}} \), we have:

\[
\frac{dE}{dt} = -\frac{k_{31}}{k_{13}} k_{11} n_1 - \frac{k_{32}}{k_{23}} k_{22} n_2 - k_{33} n_3
\]

it’s clear that

\[
\frac{dE}{dt} < 0 \quad \text{for} \quad k_{11} \geq 0 \quad \text{and} \quad k_{22} \geq 0 \quad \text{that is} \quad N_1^* \geq a \quad \text{and} \quad N_2^* \geq b. \quad (4.7)
\]

and \( \frac{dE}{dt} = 0 \) only when \( n_1 = n_2 = n_3 = 0 \). Then the equilibrium state is asymptotically stable provided that (4.7) hold.
To see the effects of dispersal on the linear stability of equilibrium state, consider the following positive definite function:

$$E = \frac{1}{2} \int_0^L (c_1 n_1^2 + c_2 n_2^2 + n_3^2) dx$$

where $c_1 = \frac{k_{31}}{k_{13}}$ and $c_2 = \frac{k_{32}}{k_{23}}$.

Then, we have

$$\frac{dE}{dt} = -\frac{k_{31}}{k_{13}} k_{11} \int_0^L n_1^2 dx - \frac{k_{32}}{k_{23}} k_{22} \int_0^L n_2^2 dx - \frac{k_{33}}{k_{23}} \int_0^L n_3^2 dx$$

Integrating by parts and using (2.6) we have

$$\frac{dE}{dt} = -\frac{k_{31}}{k_{13}} k_{11} \int_0^L n_1^2 dx - \frac{k_{32}}{k_{23}} k_{22} \int_0^L n_2^2 dx - \frac{k_{33}}{k_{23}} \int_0^L n_3^2 dx$$

$$-\frac{k_{31}}{k_{13}} \int_0^L D_1 (\frac{\partial n_1}{\partial x})^2 dx - \frac{k_{32}}{k_{23}} \int_0^L D_2 (\frac{\partial n_2}{\partial x})^2 dx - \int_0^L D_3 (\frac{\partial n_3}{\partial x})^2 dx.$$  \hspace{1cm} (4.8)

Hence, we see that for $N_1^* \geq a$ and $N_2^* \geq b$, $\frac{dE}{dt} < 0$ and the equilibrium state is asymptotically stable.

(b) If $N_1^* < a$ and $N_2^* < b$ i.e. $k_{11} < 0$ and $k_{22} < 0$.

We can find certain conditions for asymptotically stable under the conditions

$$n_i(0, t) = n_i(L, t) = 0, \quad i = 1, 2, 3$$

provided $D_i(x) \geq D_i > 0$.

In this case using Poincaree inequality in (4.8) we have

$$\frac{dE}{dt} \leq -(\frac{k_{31}}{k_{13}} k_{11} + \frac{k_{31}}{k_{13}} D_1 \frac{\pi^2}{L^2}) \int_0^L n_1^2 dx - (\frac{k_{32}}{k_{23}} k_{22} + \frac{k_{32}}{k_{23}} D_2 \frac{\pi^2}{L^2}) \int_0^L n_2^2 dx$$

$$- (k_{33} + \frac{D_3 \pi^2}{L^2}) \int_0^L n_3^2 dx.$$  \hspace{1cm} (4.9)

It is clear that $\frac{dE}{dt} < 0$ when $D_1 \geq \frac{2\alpha a_{11} L^2}{\pi^2} \frac{N_1^* (a - N_1^*)}{1 + \alpha N_1^*}$ and $D_2 \geq \frac{2\alpha a_{22} L^2}{\pi^2} \frac{N_2^* (b - N_2^*)}{1 + \alpha N_2^*}$, and $\frac{dE}{dt} = 0$ only when $n_1 = n_2 = n_3 = 0$.

Then there is stability of equilibrium state, showing that dispersal has a stabilizing effect.
(c) If \( N_1^* < a \) and \( N_2^* \geq b \) i.e. \( k_{11} < 0 \) and \( k_{22} \geq 0 \), from (4.9) we find that \( \frac{dE}{dt} < 0 \) when \( D_1 \geq \frac{2aa_{11}L^2 N_1^*(a - N_1^*)}{\pi^2} \) and \( \frac{dE}{dt} = 0 \) only when \( n_1 = n_2 = n_3 = 0 \). Similarly, if \( N_1^* \geq a \) and \( N_2^* < b \) i.e. \( k_{11} \geq 0 \) and \( k_{22} < 0 \).

Then from (4.9) we find that \( \frac{dE}{dt} < 0 \) when \( D_2 \geq \frac{2aa_{22}L^2 N_2^*(b - N_2^*)}{\pi^2} \) and \( \frac{dE}{dt} = 0 \) only when \( n_1 = n_2 = n_3 = 0 \).

And, finally there is stability of equilibrium state, showing that dispersal has a stabilizing effect.

**Theorem 4.2**

(a) If \( N_1^* \geq 2a \) and \( N_2^* \geq 2b \), the equilibrium point \( E(N_1^*, N_2^*, N_3^*) \) of system (4.1) is non-linear asymptotically stable (with and without dispersal) in the entire positive octant.

(b) If, \( a \leq N_1^* < 2a \) and \( b \leq N_2^* < 2b \), the equilibrium point \( E(N_1^*, N_2^*, N_3^*) \) of system (4.1) is non-linear asymptotically stable (with and without dispersal) in the region \( A \)

\[ A = \{ (N_1, N_2, N_3); N_1 \geq 2a - N_1^*, N_2 \geq 2b - N_2^*, N_3 > 0 \} \]

(c) If, \( N_1^* < a \) and \( N_2^* < b \), the equilibrium point \( E(N_1^*, N_2^*, N_3^*) \) of system (4.1) is not non-linear asymptotically stable without dispersal, but with constant dispersal, it does become stable in the region:

\[ B = \{ (N_1, N_2, N_3); 0 < N_1 < 2a - N_1^*, 0 < N_2 < 2b - N_2^*, N_3 > 0 \} \]

under the conditions

\[ D_1 \geq \frac{aa_{11}L^2}{\pi^2} \frac{N^3}{N_1^*(1 + \alpha N_1^*)} \quad \text{and} \quad D_2 \geq \frac{aa_{22}L^2}{\pi^2} \frac{M^3}{N_2^*(1 + \alpha N_2^*)} \]

where

\[ N = 2a - N_1^* \quad \text{and} \quad M = 2b - N_2^*. \]

**Proof**

To study the non-linear stability of the equilibrium state in the absence of dispersal, consider the positive definite function in (3.1).

Using (4.3) and choosing \( c_1 = \frac{a_{31}}{a_{13}(1 + \alpha N_1^*)} \) and \( c_2 = \frac{a_{32}}{a_{23}(1 + \alpha N_2^*)} \), then we have

\[ \frac{dE}{dt} = \frac{aa_{31}a_{11}}{a_{13}} (N - N_1) \frac{n_1^2}{s_1} + \frac{aa_{32}a_{22}}{a_{23}} (M - N_2) \frac{n_2^2}{s_2} - a_{33}n_3^2 \quad (4.10) \]
where \( N = 2a - N_1^* \) and \( M = 2b - N_2^*. \) \( (4.11) \)

Now, we get from this equation the following cases:

(a) If

\[
N \leq 0 \text { and } M \leq 0 \quad \text{i.e. } N_1^* \geq 2a \text { and } N_2^* \geq 2b. \quad (4.12)
\]

Then \( \frac{dE}{dt} < 0 \) in the positive octant: \( p = \{ (N_1, N_2, N_3); \ N_i > 0, \ i = 1, 2, 3 \} \) and the equilibrium state is non-linearly asymptotically stable in the entire positive octant provided (4.12) is satisfied.

(b) If

\[
N > 0 \text { and } M > 0 \quad \text{i.e. } 0 < N_1^* < 2a \text { and } 0 < N_2^* < 2b. \quad (4.13)
\]

Consider the subregion \( A = \{ (N_1, N_2, N_3); \ N_1 \geq N, \ N_2 \geq M, \ N_3 > 0 \} \) of the positive octant. Then \( \frac{dE}{dt} < 0 \) in \( A \), which includes the equilibrium state provided (4.12) is satisfied.

Hence, we conclude that the criteria for asymptotically stable of the equilibrium state can be written as

\[
a \leq N_1^* < 2a, \quad b \leq N_2^* < 2b \quad (4.14)
\]

which are conditions of non-linearly stable in the region \( A \).

Since, \( N_1^* \geq a \) and \( N_2^* \geq b \) are the conditions for linear stability, it is concluded that if the equilibrium state is linearly stable, it is non-linearly stable in the region \( A \) under (4.14).

To study the effect of dispersal on the non-linearly stability of the equilibrium state, consider the positive definite function in (3.3). Then using (4.3), we have

\[
\frac{dE}{dt} = \frac{\alpha a_{31}}{a_{13}} a_{11} \int_0^L (N - N_1^*) \frac{n_1^2}{s_1} \, dx + \frac{\alpha a_{32}}{a_{23}} a_{22} \int_0^L (M - N_2^*) \frac{n_2^2}{s_2} \, dx - a_{33} \int_0^L n_3^2 \, dx
\]

\[
+c_1 \int_0^L \frac{n_1}{N_1^* + n_1} \frac{\partial}{\partial x} (D_1 \frac{\partial n_1}{\partial x}) \, dx + c_2 \int_0^L \frac{n_2}{N_2^* + n_2} \frac{\partial}{\partial x} (D_2 \frac{\partial n_2}{\partial x}) \, dx
\]

Integrating by parts and using (2.6), we have

\[
\frac{dE}{dt} = \frac{\alpha a_{31}}{a_{13}} a_{11} \int_0^L (N - N_1^*) \frac{n_1^2}{s_1} \, dx + \frac{\alpha a_{32}}{a_{23}} a_{22} \int_0^L (M - N_2^*) \frac{n_2^2}{s_2} \, dx - a_{33} \int_0^L n_3^2 \, dx
\]

\[
-c_1 \int_0^L \frac{N_1^* D_1}{(N_1^* + n_1)^2} (\frac{\partial n_1}{\partial x})^2 \, dx - c_2 \int_0^L \frac{N_2^* D_2}{(N_2^* + n_2)^2} (\frac{\partial n_2}{\partial x})^2 \, dx
\]

\[
- \int_0^L \frac{N_3^* D_3}{(N_3^* + n_3)^2} (\frac{\partial n_3}{\partial x})^2 \, dx. \quad (4.15)
\]
Now, we get from (4.15) the following cases:

(a) If $N \leq 0$ and $M \leq 0$ i.e. $N_1^* \geq 2a$ and $N_2^* \geq 2b$. The equilibrium state $E(N_1^*, N_2^*, N_3^*)$ which is asymptotically stable in the entire octant $p$ remains so with dispersal also.

(b) If $N > 0, N_1^* \geq N$ and $M > 0, N_2^* \geq M$ i.e. $a \leq N_1^* < 2a$ and $b \leq N_2^* < 2b$.

The equilibrium state $(N_1^*, N_2^*, N_3^*)$ is asymptotically stable in subregion

$$A = \{(N_1, N_2, N_3); \ N_1 \geq N, \ N_2 \geq M, \ N_3 > 0\}$$

of the positive octant with dispersal as well.

(c) If $N > 0, N_1^* < N$ and $M > 0, N_2^* < M$ i.e. $a \leq N_1^* < 2a$ and $b \leq N_2^* < 2b$, does not hold in the region

$$B = \{(N_1, N_2, N_3); \ 0 < N_1 < 2a - N_1^*, \ 0 < N_2 < 2b - N_2^*, \ N_3 > 0\}.$$ 

Since $N_1 < N$ and $N_2 < M$, then

\[
\frac{dE}{dt} \leq \frac{a_31a_{11}}{a_{13}} \int_{0}^{L} (N - N_1) \frac{n_1^2}{s_1} dx + \frac{a_32a_{22}}{a_{23}} \int_{0}^{L} (M - N_2) \frac{n_2^2}{s_2} dx
\]

\[
- a_{33} \int_{0}^{L} n_3^2 dx - \frac{c_1N_1^*}{N_2} \int_{0}^{L} D_1 \frac{\partial n_1}{\partial x}^2 dx
\]

\[
- \frac{c_2N_2^*}{M^2} \int_{0}^{L} D_2 \frac{\partial n_2}{\partial x}^2 dx - \int_{0}^{L} \frac{D_3 N_3}{(N_3^* + n_3)^2} \frac{\partial n_3}{\partial x}^2 dx.
\]

(4.16)

We can find certain conditions for asymptotically stable under conditions (2.6ii) provided $D_i(x) \geq 0, \ i = 1, 2, 3$.

In this case using poincaree inequality in (4.16), we have

\[
\frac{dE}{dt} \leq \frac{a_31a_{11}}{a_{13}} \int_{0}^{L} (N - N_1) \frac{n_1^2}{s_1} dx + \frac{a_32a_{22}}{a_{23}} \int_{0}^{L} (M - N_2) \frac{n_2^2}{s_2} dx
\]

\[
- a_{33} \int_{0}^{L} n_3^2 dx - \frac{c_1D_1}{N_2L^2} \int_{0}^{L} n_1^2 dx - \frac{c_2D_2}{M^2L^2} \int_{0}^{L} n_2^2 dx
\]

\[- \int_{0}^{L} \frac{D_3 N_3}{(N_3^* + n_3)^2} \frac{\partial n_3}{\partial x}^2 dx.
\]

(4.17)

Then we can show that $\frac{dE}{dt} < 0$ if

\[
D_1 \geq \frac{a_{11}L^2}{\pi^2} \frac{N^3}{N_1^*(1 + \alpha N_1^*)}, \quad D_2 \geq \frac{a_{22}L^2}{\pi^2} \frac{M^3}{N_2^*(1 + \alpha N_2^*)}.
\]

(4.18)
Showing the asymptotically stable of the equilibrium state in the region

\[ B = \{(N_1, N_2, N_3); \ 0 < N_1 < 2a - N_1^*, \ 0 < N_2 < 2b - N_2^*, \ N_3 > 0\} \]

provided (4.18) hold.

Thus it is concluded that dispersal has stabilizing effect even on the non-linear stability of the equilibrium state of system (4.1).

4.1 Corollary

If \( N_1^* \geq 2a \) and \( b \leq N_2^* < 2b(a \leq N_1^* < 2a \) and \( N_2^* \geq 2b \)\) the equilibrium point \( E(N_1^*, N_2^*, N_3^*) \) of system (4.1) is non-linear asymptotically stable

\[ A = \{(N_1, N_2, N_3); \ N_1 > 0, \ N_2 \geq 2b - N_2^*, \ N_3 > 0\} \]

\[ A = \{(N_1, N_2, N_3); \ N_1 \geq 2a - N_1^*, \ N_2 > 0, \ N_3 > 0\} \]

4.2 Corollary

If \( N_1^* < a \) and \( N_2^* \geq 2b(N_1^* \geq 2a \) and \( N_2^* < b \)\) the equilibrium point \( E(N_1^*, N_2^*, N_3^*) \) of system (4.1) is not non-linear asymptotically stable without dispersal, but with constant dispersal it does become stable in the region

\[ B = \{(N_1, N_2, N_3); \ 0 < N_1 < 2a - N_1^*, \ N_2 > 0, \ N_3 > 0\} \]

\[ B = \{(N_1, N_2, N_3); \ N_1 > 0, \ 0 < N_2 < 2b - N_2^*, \ N_3 > 0\}. \]

4.3 Corollary

If \( N_1^* > a \) and \( b \leq N_2^* < 2b(a \leq N_1^* < 2a \) and \( N_2^* < b \)\) the equilibrium point \( E(N_1^*, N_2^*, N_3^*) \) of system (4.1) is not non-linear asymptotically stable without dispersal, but with constant dispersal it does become stable in the region

\[ c = \{(N_1, N_2, N_3); \ 0 < N_1 < 2a - N_1^*, \ N_2 \geq 2b - N_2^*, \ N_3 > 0\} \]

\[ c = \{(N_1, N_2, N_3); \ N_1 \geq 2a - N_1^*, \ 0 < N_2 < 2b - N_2^*, \ N_3 > 0\}. \]

References


On stability of three interacting species


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