Logharmonic Surfaces

Z. AbdulHadi and Y. Abu Muhanna

Department of Mathematics
American University of Sharjah
Sharjah, Box 26666, UAE
zahadi@aus.edu, ymuhanna@aus.edu

Abstract. In this paper, we consider the class of univalent logharmonic mappings defined on the unit disc $U$. We shall address the issue which of the logharmonic maps produce isothermal surfaces. A classification of these mappings will be included.

Mathematics Subject Classification: Primary 30C35, 30C45; Secondary 35Q30

Keywords: logharmonic, univalent, isothermal, minimal surfaces, Gauss Curvature

1. Introduction

Throughout,

$$\Phi : \Phi(x, y) = (\Phi_1, \Phi_2, \Phi_3)$$

denote a $C^2$ surface in $R^3$ where $x + iy = z \in U = \{z : |z| < 1\}$. If $k_1$ and $k_2$ are the maximum and minimum of the normal curvatures at a point on $\Phi$, then the mean curvature $H = \frac{k_1 + k_2}{2}$ and the Gauss curvature $K = k_1k_2$, (more details are in [6], [8], [9]).

We call the surface $\Phi$ isothermal if the mapping $\Phi(x, y)$ of the parametric domain $U$ onto the surface is locally conformal, so that angles between curves on the surface are equal to the angles between the corresponding curves in the parameter plane. Analytically, see [6], [9], this condition is equivalent to the conditions

$$\Phi_x.\Phi_x = \Phi_y.\Phi_y = \lambda^2 > 0$$
$$\Phi_x.\Phi_y = 0,$$

(1.1)
or equivalently,
\[ \det \Phi_{ij} = \det \begin{pmatrix} \Phi_x \Phi_y & \Phi_x \Phi_y \\ \Phi_x \Phi_y & \Phi_x \Phi_y \end{pmatrix} = \lambda^4. \]

It is also known that, when \( \Phi \) is isothermal,
\[ \Delta \Phi = 2\lambda^2 H, \]
\[ K = -\frac{\Delta \log \lambda}{\lambda^2}. \]

where \( \Delta \) is the Laplacian, see [6], [9].

Hence, as a minimal surface is equivalent to the condition \( H = 0 \), it follows that an isothermal surface is minimal if and only if the components of \( \Phi \) are harmonic, (for more details see [6], [9]). If, in addition, \( \Phi \) is non-parametric (graph) then the projection is a harmonic univalent map and conversely any harmonic univalent map produces a non-parametric minimal surface, [6]. For more information on harmonic univalent maps and minimal surfaces, see [5], [7].

It is clear from the preceding paragraph that a logharmonic map does not produce a minimal surface.

in this paper we address the issue of whether a logharmonic map produces an isothermal surface. Unfortunately we find out that not every logharmonic map produces an isothermal surface but some do.

By a logharmonic surface we mean

\[ \Phi_L : (\Re h(z)\overline{g(z)}, \Im h(z)\overline{g(z)}, S), \]

where \( h \) and \( g \) are analytic and when (1.2) satisfies (1.1), \( \Phi_L \) is called isothermal logharmonic surface.

In this article, we shall classify the isothermal logharmonic surfaces.

In Theorem 1, we give a necessary and sufficient condition for \( \Phi_L \) to be isothermal. In addition \( \Phi_L \) is non-parametric when the corresponding logharmonic map is univalent.

In Lemma 5 and Lemma 6, we write \( H, K \) for isothermal logharmonic surface, in terms of the logharmonic map. Finally we give two examples of non-parametric isothermal logharmonic surfaces.

Next we provide the definition and some information about logharmonic maps. Let \( B \) denote the set of all analytic functions \( a \) defined on the unit disk \( U \) having the property that \( |a(z)| < 1 \) for all \( z \in U \). A logharmonic mapping defined on the unit disk \( U \) is a solution of the nonlinear elliptic partial differential equation

\[ \frac{\partial^2 a}{\partial z^2} + \frac{\partial^2 a}{\partial \overline{z}^2} = 0. \]
where the second dilatation function \( \alpha \in \mathbb{C} \). Because \( \alpha \in \mathbb{C} \), the Jacobian

\[
\frac{\bar{f}_z}{f} = a \frac{f_z}{f},
\]

is positive and hence, non-constant logharmonic mappings are sense-preserving and open on \( U \). If \( f \) is a non-constant logharmonic mapping of \( U \) and vanishes at \( z = 0 \) but has no other zeros in \( U \), then \( f \) admits the following representation

\[
f(z) = z^m |z|^{2\beta} H(z) g(z),
\]

where \( m \) is a nonnegative integer, \( \Re{\beta} > -1/2 \) and, \( H \) and \( g \) are analytic functions in \( U \) with \( g(0) = 1 \) and \( H(0) \neq 0 \) (cf.[1]). The exponent \( \beta \) in (1.3) depends only on \( a(0) \) and can be expressed by

\[
\beta = \frac{a(0)}{1 - |a(0)|^2}.
\]

Note that \( f(0) \neq 0 \) if and only if \( m = 0 \) and that a univalent logharmonic mapping on \( U \) vanish at the origin if and only if \( m = 1 \). Thus, a univalent logharmonic mappings on \( U \) which vanishes at the origin will be of the form

\[
f(z) = z|z|^{2\beta} H(z) g(z),
\]

where \( \Re{\beta} > -1/2 \) and \( 0 \notin (Hg)(U) \). More details can be found in [1, 2, 3, 4].

In this article, we shall consider the subclass \( S_{\Phi}^0 \), of all univalent logharmonic maps with \( \beta = 0 \) and

\[
f(z) = h(z) g(z),
\]

where \( h(z) = z H(z) \).

2. Isothermal Logharmonic Surfaces

The main result of this section is Theorem 1. The following series of lemmas are steps toward Theorem 1.

**Lemma 1.** Let \( f(z) = h(z) g(z) \in S_{\Phi}^0 \) and let \( \Phi_L \) be as in (1.2), is isothermal then we have

\[
S_{z}^2 = -h g' g', \quad \quad \quad S_{\bar{z}}^2 = -h g' g'.
\]
Proof. Direct calculations give

\[
\Phi(z) = (\text{Re}(h'\bar{y} + h\bar{y}' + S_x), S_x)
\]
\[
\Phi(z) = (\text{Im}(h'\bar{y} - ih\bar{y}' + S_y), S_y),
\]
\[
\Phi_x, \Phi_y = \frac{|h'\bar{y} + h\bar{y}'|^2 + |S_x|^2}{2}
\]
\[
\Phi_x, \Phi_y = \frac{|h\bar{y} - h\bar{y}'|^2 + |S_y|^2}{2},
\]
\[
\Phi_x, \Phi_y = -\text{Re}(h'\bar{y} + h\bar{y}') \text{Im}(h\bar{y} - h\bar{y}') + \text{Im}(h'\bar{y} + h\bar{y}') \text{Re}(h\bar{y} - h\bar{y}') + S_x S_y.
\]

The condition in (1.1), \(\Phi_x, \Phi_y = 0\) implies that

\[
(2.1) \quad |S_x|^2 - |S_y|^2 = -4 \text{Re} hgh'g'.
\]

The other condition in (1.1), \(\Phi_x, \Phi_y = 0\) implies that

\[
(2.2) \quad S_x S_y = 2 \text{Im} hgh'g'.
\]

Hence, (2.2) and (2.3) give

\[
S_x^2 = \left(\frac{S_x + iS_y}{2}\right)^2 = -\text{Re} hgh'g' + i \text{Im} hgh'g' = -hgh'g'.
\]

Similarly,

\[
S_y^2 = -hgh'g'.
\]

Corollary 1. Let \(f(z) = h(z)g(z) \in S_{Lh}^0\) such that \(f\) and \(\Delta f\) are non-vanishing in the disk \(U\). If \(\Phi_L\), in (1.2), is isothermal then

\[
S_x(z) = -2 \text{Im} \sqrt{hgh'g'}
\]
\[
S_y(z) = -2 \text{Re} \sqrt{hgh'g'}.
\]

Proof. (2.1) implies \(S_x = i\sqrt{hgh'g'}\) and \(S_y = -i\sqrt{hgh'g'}\). Then as \(S_x = (S_x + S_y)\) and \(S_y = i(S_x - S_y)\) the result follows. \(\square\)

The following lemma gives an exactness condition for the system (2.4)

Lemma 2. Let \(\Phi_L\) and \(f\) be as in Corollary 1, then \(S_{xy} = S_{yx}\) if and only if \(\text{Re} \frac{\sqrt{hgh'}}{\sqrt{h}} \left[\frac{g}{g} + \frac{h}{g}\right] = 0\) in \(U\).

Proof. From Corollary 1, we have

\[
S_{xy} = -2 \text{Im} \left[\sqrt{hgh'} \frac{i}{z} + \sqrt{hgh'} \frac{-i}{z}\right]
\]
\[
= -2 \text{Re} \left[\sqrt{hgh'} \frac{i}{z} - \sqrt{hgh'} \frac{-i}{z}\right].
\]
\[
S_{yx} = -2 \text{Re} \left[\sqrt{hgh'} \frac{i}{z} + \sqrt{hgh'} \frac{-i}{z}\right].
\]

The condition \(S_{xy} = S_{yx}\) implies that
Logharmonic surfaces

\[ \text{Re} \left[ \sqrt{h'g'} \left( \sqrt{hg} \right)^{-1} \right] = 0. \]

Hence

\[
\text{Re} \left[ \sqrt{h'g'} \left( \sqrt{hg} \right)^{-1} \right] = \frac{1}{2} \text{Re} \left[ \frac{\sqrt{h'g'}}{\sqrt{hg}} (h'g + hg') \right]
= \frac{1}{2} |h'g'| \text{Re} \left[ \frac{g'}{g} + \frac{h}{h'} \right] = 0.
\]

The converse follows by reversing the steps.  \( \square \)

**Corollary 2.** Let \( \Phi_L \) and \( f \) be as in Corollary 1, then

\[ \text{Re} \left[ \frac{\sqrt{h'g'}}{\sqrt{hg}} \left( \frac{g}{g'} + \frac{h}{h'} \right) \right] = 0 \]

is equivalent to the dilatation \( a(z) \) is constant and

\[-1 \leq a < 0. \]

**Proof.** \( \text{Re} \left[ \frac{\sqrt{h'g'}}{\sqrt{hg}} \left( \frac{g}{g'} + \frac{h}{h'} \right) \right] = 0 \) implies that \( \frac{\sqrt{h'g'}}{\sqrt{hg}} \left( \frac{g}{g'} + \frac{h}{h'} \right) = it. \)

Since \( \frac{\bar{f}}{f} = a \frac{\bar{f}}{f} \) or \( a = \frac{hg'}{gh'} \) we have

\[ \frac{\sqrt{h'g'}}{\sqrt{hg}} \frac{g}{g'} [1 + a] = it, \]

which leads to

\[ \frac{1}{\sqrt{a}} [1 + a] = it, \]

\[ a - it\sqrt{a} + 1 = 0. \]

Hence \( a = -\frac{1}{4} \left[ t \pm \sqrt{t^2 + 4} \right]. \)

i) If \( a = -\frac{1}{4} \left[ t + \sqrt{t^2 + 4} \right] \) and \( t < 0 \) then \( -1 < a < 0 \)

and

ii) if \( a = -\frac{1}{4} \left[ t - \sqrt{t^2 + 4} \right] \) and \( t > 0 \) then \( -1 < a < 0. \)

Conversely, the condition \( -1 < a < 0 \) implies that \( \text{Re} \left[ \frac{\sqrt{h'g'}}{\sqrt{hg}} \left( \frac{g}{g'} + \frac{h}{h'} \right) \right] = \text{Re}(1 + a) = 0. \)

Following is the main theorem
Theorem 1. Let \( f(z) = h(z)g(z) \in S^0_{Lh} \) and let \( \Phi_L \) be as in (1.2), then 
\( \Phi_L \) is isothermal if and only if 
\[
\Phi_L = (\Re h^a, \Im h^\alpha, -\sqrt{-a} |h|^{1+a}) \text{.}
\]
where \( a \) is a constant with \(-1 < a < 0\) and \( h \) is a non-vanishing analytic function.

Proof. If \( \Phi_L \) is isothermal then Lemma 2 and Corollary 2 imply that 
\[
S_x = -2\frac{\sqrt{-a}}{|h|^a} \Re (h'h) \text{, } S_y = 2\frac{\sqrt{-a}}{|h|^a} \Im (h'h) \text{.}
\]
Hence 
\[
S_x = -\frac{\sqrt{-a}}{|h|^a} (h'h) \text{ and } S_y = -\frac{\sqrt{-a}}{|h|^a} (\bar{h}h) \text{.}
\]
Integration gives the result. Clearly the converse follows. \( \Box \)

Corollary 3. \( \Phi_L \) in Theorem 1 is a non-parametric surface if the logharmonic map \( h^\alpha \) is univalent.

As a consequence of Theorem 1, we have

Lemma 3. Let \( \Phi_L \) be isothermal. Then
\[
\lambda^2 = \Phi_{Lx} \cdot \Phi_{Lx} \text{.}
\]
(2.6) 
\[
= |h^a|^{-1} \left[ |h'h + ah\bar{h}h|^2 + (-a)(1+a)^2 |h^a|^{-1} \left| \Re h'^2 \right| \right] \text{.}
\]

Proof. We have
\[
\Phi_{Lx} = h'h + ah\bar{h}h \text{.}
\]
Write \( \Phi_3 = -\sqrt{-a} h^\alpha \bar{h}^{\alpha-1} \). Then
\[
\Phi_{3x} = -\sqrt{-a} \left[ \frac{1+a}{2} h^{\alpha-1} h' + h^{\alpha-1} h' + \frac{1}{2} h^{\alpha-2} h'h^{\alpha-1} - \frac{1}{2} h^{\alpha-2} h'h^{\alpha-1} \right] =
-\sqrt{-a} \left[ \frac{1+a}{2} |h^a|^{-1} |h'h + \bar{h}h| = -\sqrt{-a} \frac{1+a}{2} |h^a|^{-1} \left| 2 \Re h'h \right| \right] \text{.}
\]
Hence,
\[
\lambda^2 = \Phi_{Lx} \cdot \Phi_{Lx}
= |h'h + ah\bar{h}h|^2 + (-a)(1+a)^2 |h^a|^{-1} \left| \Re h'^2 \right| \text{.}
\]

\( \Box \)

Lemma 4. Let \( \Phi_L \) be isothermal. Then
\[
\Delta \Phi_L = \Phi_{Lz} = 4|h'|^2 \left[ ah^{a-1}, -\sqrt{-a} \left( \frac{a+1}{2} \right) |h|^{a-1} \right] \text{,}
\]
(2.7) 
\[
\| \Delta \Phi_L \| = 4|h'|^2 |h|^{a-1} \sqrt{a^2 + (-a)(1+a)^4} \frac{16}{16} \text{.}
\]

Proof. By direct calculations, we have
\[
\Phi_{Lz} = a|h'|^2 h^{a-1}, -\sqrt{-a} \left( \frac{a+1}{2} \right)^2 |h'|^2 |h|^{a-1} = |h'|^2 \left[ ah^{a-1}, -\sqrt{-a} \left( \frac{a+1}{2} \right)^2 |h|^{a-1} \right] \text{.}
\]

The relations (2.6) and (2.8) give
Lemma 5. Let $\Phi_L$ be isothermal. Then mean curvature

$$H = \pm \frac{\| \Delta \Phi_L \|}{2\lambda^2}$$

$$= \pm \frac{2 |h^{1-a}| |h'|^2 \sqrt{a^2 + (-a)(1+a)^2}}{\left[ |h'h + ahh'|^2 + (-a)(1+a)^2 |h^{a-1}|^2 \left[ \text{Re} hh' \right]^2 \right]}.$$

Lemma 6. Let $\Phi_L$ be isothermal. Then the Gauss curvature

$$K = -\frac{\Delta \log \lambda^2}{2\lambda^2} = -\frac{\beta \bar{z} \beta - \beta \bar{z} \beta}{2\beta^4},$$

where $\beta = \lambda^2 = |h'h + ahh'|^2 + (-a)(1+a)^2 |h^{a-1}|^2 \left[ \text{Re} hh' \right]^2$.

Remark 1. The corresponding minimal surface to (2.5) is

$$\Psi_H = (\text{Re}(1+a) \log h, \text{Im}(1-a) \log h, 2 \text{Im} \sqrt{\alpha} \log h)$$

$$= (\text{Re}(1+a) \log h, \text{Im}(1-a) \log h, 2 \text{Re} \sqrt{-a} \log h)$$

$$= (\text{Re}(1+a) \log h, \text{Im}(1-a) \log h, 2 \sqrt{-a} \log |h|).$$

Exponentiation of $\Psi_H$ and compare to

$$\Phi_L = (\text{Re} hh^a, \text{Im} hh^a, -\sqrt{-a} |h|^{1+a}),$$

we conclude that $\exp \Psi_H$ is similar to $\Phi_L$ except for the heights, because $\exp 2\sqrt{-a} \log |h| = |h|^{2\sqrt{-a}} \neq -\sqrt{-a} |h|^{1+a}.$

Next we give two examples of non-parametric isothermal logharmonic surfaces.

Example 1. The function $f(z) = \exp(z - \bar{z}/2)$ is univalent logharmonic with respect to $a(z) = -1/2$. The Gauss curvature of the corresponding surface $\Phi_L$ at the origin is equal to $-\frac{108}{25}$. Figure 1 represents the images under $f$ of concentric circles in $U$, while Figure 2 is the graph of $\Phi_L$. 
Example 2. The function $f(z) = \left(\frac{1 + \bar{z}}{1 - \bar{z}}\right)\left(\frac{1 - z}{1 + z}\right)^{1/2}$ is univalent logharmonic with respect to $a(z) = -1/2$. The Gauss of the corresponding surface $\Phi_L$ at the origin is equal to $-\frac{67}{256}$. Figure 3 represents the images under $f$ of concentric circles in $U$, while Figure 4 is the graph of $\Phi_L$.

Remark 2. a) $\lambda^2|dz|$ defines a semimetric. In Example 1 the Gauss curvature at 0 is less than $-1$, hence the corresponding metric $\lambda^2|dz|$ is less than the hyperbolic metric in some neighborhood of 0.

b) In both examples the curvature is negative,

Problem 1. Is the Gauss curvature always negative for isothermal logharmonic surfaces?
REFERENCES


Received: March, 2012