Remark on Uniform Attractor for the 3D Non-autonomous Brinkman-Forchheimer Equation

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Abstract

In this paper, we show the existence result of uniform attractors for the 3D non-autonomous Brinkman-Forchheimer equation.

Keywords: processes, upper semicontinuity, uniform attractor

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary $\partial \Omega$. We consider the large time behavior for a non-autonomous 3D Brinkman-Forchheimer equation:

\begin{align}
& u_t - \nu \Delta u + \alpha u + \beta |u|u + \gamma |u|^2u + \nabla p = \sigma(t, x), \quad (1) \\
& \text{div} u = 0, \quad x \in \Omega, \quad t \in [\tau, +\infty), \quad (2) \\
& u(t, x)|_{\partial \Omega} = 0, \quad t \in [\tau, +\infty), \quad (3) \\
& u(\tau, x) = u_\tau(x), \quad x \in \Omega, \quad \tau \geq 0. \quad (4)
\end{align}

Here $(x, t) \in \Omega \times [\tau, +\infty)$, $u = u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ is the velocity vector field, $p$ is the pressure, $\nu > 0$ is the Brinkman kinematic viscosity.
Let us recall some known results for the Brinkman-Forchheimer equation. [3] and [10] investigated the physics significance and theoretical development of the 3D Brinkman-Forchheimer equation respectively, when the Reynolds number for fluid is low, [3] shows the Darcy’s Law also holds except some corrections. For the autonomous Brinkman-Forchheimer equation, it was shown in [1], [2], [4] and [6] that the Brinkman-Forchheimer equation has global solutions and the solution is continuous dependence on the coefficients, moreover, they also derived the convergence of corresponding solutions as coefficients tend to zero. [8], [5] and [9] obtained the existence of global attractor of Brinkman-Forchheimer equation by different methods. However, there are fewer results for the non-autonomous case, [7] deduced the existence of $\mathcal{D}$-pullback attractors for 3D non-autonomous Brinkman-Forchheimer equation by establishing the $\mathcal{D}$-pullback asymptotical compactness of $\theta$-cocycle recently, [11] gave the existence of uniform attractor of 3D non-autonomous Brinkman-Forchheimer equation by continuous method.

In this paper, we shall describe our main result and some simple proof of [11].

2 Main Results

The family of functions $L^2_{loc}(R; H)$ denote a local Bochner integration function class, $L^2_b(R; H)$ denotes all translation bounded functions which satisfies $\sup_{t \in R} \int_{t}^{t+1} \|\sigma(s, x)\|^2_H ds < +\infty$ for all $\sigma \in L^2_{loc}(R; H)$, i.e., $\sigma$ is translation bounded in $L^2_{loc}(R; H)$. Obviously, $L^2_b(R; H) \subset L^2_{loc}(R; H)$.

The problem (1)-(2) can be written as an abstract form

\begin{align}
  u_t + \nu Au + \alpha u + B(u) &= \sigma(t, x), \\
  \text{div} u &= 0,
\end{align}

(5) (6)

where the pressure $p$ has disappeared by force of the application of the Leray-Helmholtz projection $P$, $B(u) = PF(u)$, $F(u) = \beta|u|u + \gamma|u|^2u$. Clearly, system (5)-(6) is equivalent to (1)-(2).

The existence and uniqueness of global solution for (1)-(4) can be derived by standard method as in [9], [2] or [4]:

**Theorem 2.1** Assume $\sigma \in L^2_{loc}(R, H)$, $u_\tau \in H$, then problem (1)-(4) possesses a unique global solution $u(t, x)$ which satisfies

\begin{align}
  u \in C([\tau, +\infty); H) \cap L^2(\tau, T; V) \cap L^4(\tau, T; (L^4(\Omega))^3). \tag{7}
\end{align}
Moreover, we choose a non-autonomous external force \( \sigma_0(t, x) \in L^2_b(R, H) \) and fixed, the global solution \( u(t, x) \) generates a process \( \{U_\sigma(\tau, t)\} \) (\( \tau, \sigma \in \Sigma \)) which is continuous with respect to \( u_\tau \), where \( \sigma \) is a symbol which belong to the symbol space \( \Sigma = \mathcal{H}(\sigma_0) = \{\sigma_0(s + h) | h \in R\} \) \( L^2_{loc}(R, H) \). \( E \) means the closure in the topology \( E \).

**Theorem 2.2** Assume that \( u_\tau \in H, \sigma \in \Sigma \subset L^2_{loc}(\tau, +\infty; H) \), then the family of processes \( \{U_\sigma(t, \tau), t \geq \tau \in R\} \) \((\sigma \in \mathcal{H}(\sigma_0)) \) generated by the global solution of problem (1)-(4) possesses a uniform (w.r.t. \( \sigma \in \Sigma = \mathcal{H}(\sigma_0) \)) attractor \( \mathcal{A}_{\mathcal{H}(\sigma_0)} = \mathcal{A}_{\Sigma} \) in \( H \).

### 3 Discussion

Choosing an arbitrary non-autonomous force \( \sigma_0(x, t) \in L^2_b(R, H) \) and then fixed, i.e.,

\[
\sup_{t \in R} \int_t^{t+1} \|\sigma_0(s)\|^2_H ds < +\infty
\]

taking \( \Sigma = \mathcal{H}(\sigma_0) \) (Defined in Theorem 2.1) as the symbol space of problem (1)-(4), \( \forall \sigma \in \Sigma \) is called the symbol of the system (1)-(4). Obviously, \( \mathcal{H}(\sigma_0) \) is strictly invariant under the acting of the translation semigroup \( \{S(h)\}_{h \geq 0} \), i.e., \( S(h)\mathcal{H}(\sigma_0) \equiv \mathcal{H}(\sigma_0) \) for all \( h \geq 0 \).

From Theorem 2.1, the global solution generates processes class \( \{U_\sigma(t, \tau), t \geq \tau, \tau \in R\} \), \( \sigma \in \mathcal{H}(\sigma_0) \), i.e., \( U_\sigma(t, \tau)u_\tau = u(t) \), where \( u(t) \) is the solution of problem (1)-(4) with symbol \( \sigma \in \Sigma \) and initial data \( u_\tau \in H \).

**Lemma 3.1** Let the external force \( \sigma \in \Sigma, u_\tau \in H \), then the process has a bounded uniform (w.r.t. \( \sigma \in \mathcal{H}(\sigma_0) \)) absorbing set \( B_0 \) in \( H \), where \( B_0 = \{u \in H : \|u\|_H \leq C\|\sigma_0\|_{L^2_b(R, H(\Omega))} \doteq \rho\} \) is a bounded set in \( H \).

**Proof.** Multiplying (1) with \( u \) and integrating on \( \Omega \), by the Young inequality we conclude

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|\nabla u\|^2 + \alpha \|u\|^2 + \beta \|u\|^3_{L^3} + \gamma \|u\|^4_{L^4} = \int_\Omega \sigma(t, x) u dx \\
\leq \frac{\alpha}{2} \|u\|^2 + \frac{2\|\sigma\|^2}{\alpha}, \tag{8}
\]
	hen integrating over \([\tau, t] \), it follows

\[
\|u\|^2 + \int_{\tau}^{t} (2\nu \|\nabla u\|^2 + \alpha \|u\|^2 + 2\beta \|u\|^3_{L^3} + 2\gamma \|u\|^4_{L^4}) ds \\
\leq \frac{4}{\alpha} \int_{\tau}^{t} \|\sigma(s)\|^2 ds + \|u_\tau\|^2, \tag{9}
\]
hence,
\[
\|u\|^2 + \int_{\tau}^{t} (2\lambda_1 \nu + \alpha) \|u\|^2 ds \leq \frac{4}{\alpha} \int_{\tau}^{t} \|\sigma(s)\|^2 ds + \|u_\tau\|^2, \tag{10}
\]
where \(\lambda_1\) is the first eigenvalue in the Poincaré inequality.

By Gronwall’s inequality, we derive
\[
\|u(t)\|^2 \leq \|u_\tau\|^2 e^{-(2\lambda_1 \nu + \alpha)(t - \tau)} + \frac{4}{\alpha} \int_{\tau}^{t} e^{(2\lambda_1 \nu + \alpha)(s - t)} \|\sigma(s)\|^2 ds \\
\leq \|u_\tau\|^2 e^{-(2\lambda_1 \nu + \alpha)(t - \tau)} + \frac{4}{\alpha} \left( \int_{t-1}^{t} e^{(2\lambda_1 \nu + \alpha)(s - t)} \|\sigma(s)\|^2 ds \\
+ \int_{t-2}^{t-1} e^{(2\lambda_1 \nu + \alpha)(s - t)} \|\sigma(s)\|^2 ds + \cdots \right) \\
\leq \|u_\tau\|^2 e^{-(2\lambda_1 \nu + \alpha)(t - \tau)} + \frac{4}{\alpha} \left( \int_{t-1}^{t} \|\sigma(s)\|^2 ds \\
+ e^{-(2\lambda_1 \nu + \alpha)} \int_{t-2}^{t-1} \|\sigma(s)\|^2 ds + e^{-(2\lambda_1 \nu + \alpha)} \int_{t-3}^{t-2} \|\sigma(s)\|^2 ds + \cdots \right) \\
\leq \|u_\tau\|^2 e^{-(2\lambda_1 \nu + \alpha)(t - \tau)} + \frac{4}{\alpha} \left( 1 + \frac{1}{(2\lambda_1 \nu + \alpha)} \right) \|\sigma_0\|^2_{L^2_b(R;H)} \\
\leq \|u_\tau\|^2 e^{-(2\lambda_1 \nu + \alpha)(t - \tau)} + \frac{4}{\alpha} \left( 1 + \frac{1}{(2\lambda_1 \nu + \alpha)} \right) \|\sigma_0\|^2_{L^2_b(R;H)}, \tag{11}
\]
choosing \(\|u_\tau\|^2 e^{-(2\lambda_1 \nu + \alpha)(t - \tau)} \leq \frac{4}{\alpha} \left( 1 + \frac{1}{(2\lambda_1 \nu + \alpha)} \right) \|\sigma_0\|^2_{L^2_b(R;H)}\); then there exists a time \(t_0 = t_0(\alpha, \lambda, \|\sigma_0\|^2_{L^2_b(R;H)})\) such that \(B_0 = \{ u : \|u\|^2 \leq \rho^2 \}\), where \(\rho^2 = \frac{8}{\alpha} \left( 1 + \frac{1}{(2\lambda_1 \nu + \alpha)} \right) \|\sigma_0\|^2_{L^2_b(R;H)}\) i.e., \(B_0\) is the uniformly (w.r.t. \(\sigma \in \Sigma\)) absorbing ball for the process \(U_\sigma(t, \tau)\) in \(H\).

**Lemma 3.2** For any \(\sigma \in \mathcal{H}(\sigma_0), u_\tau \in H\), the family of processes \(\{U_\sigma(t, \tau), t \geq \tau \in R\}\), \(\sigma \in \mathcal{H}(\sigma_0)\) defined on \(H\), corresponding to equations (1)-(4) is uniformly (w.r.t. \(\sigma \in \Sigma\)) asymptotically compact in \(H\).

**Proof.** Assume that \(\{u^n_\tau\}\) is a bounded sequence in \(H\), \(\{\sigma^n\} \subset \mathcal{H}(\sigma_0)\) and \(\{t_n\} \in (\tau, +\infty), t_n \to +\infty\) as \(n \to +\infty\).

From the proof of the existence of uniformly absorbing set, we see that for any fixed \(\tau \in R\), there exists a time \(T_0 = T_0(\rho, \tau)\) dependent on radius \(\rho\) of the absorbing ball and \(\tau\), such that for all \(t_n \geq T_0\), \(\{U_{\sigma^n}(t_n, \tau)u^n_\tau\} \subseteq B_0\), \(B_0\) is defined in Lemma 3.1.

From Theorem 2.1, the sequence \(\{U_{\sigma^n}(t_n, \tau)u^n_\tau\}\) is weakly precompact in \(H\) and we have
\[
U_{\sigma^n}(t_n, \tau)u^n_\tau \rightharpoonup u \text{ weakly in } H \text{ as } n \to +\infty \tag{12}
\]
for some \(u \in H\) and some subsequence (still denoted by) \(U_{\sigma^n}(t_n, \tau)u^n_\tau\).
Similarly, for each $T > 0$ and $t_n \geq T_0 + T$,
\[
u^n_T = U\sigma^n(t_n - T, \tau)u^n_T \rightarrow u_T \text{ weakly in } H \text{ as } n \rightarrow +\infty
\] (13)
for some $u_T \in H$.
Noting that the translation semigroup $\{S(h) : h \geq 0\}$ satisfies
\[
U_{S(h)\sigma}(t, \tau) = U_\sigma(t + h, \tau + h), \quad \forall \ h \geq 0, \ t \geq \tau \in \mathbb{R}, \ \forall \ \sigma \in \mathcal{H}(\sigma_0),
\] (14)
we see that
\[
U_\sigma(t_n, \tau) = U_\sigma(T, 0)U_\sigma(n - T, \tau), \ t_n - T \geq \tau.
\] (15)
Let $\sigma^n_T = S(t_n - T)\sigma^n$, by (13)–(15), we have
\[
U_\sigma(t_n, \tau)u^n_T = U_\sigma(T, 0)U_\sigma(n - T, \tau)u^n_T, \ t_n - T \geq \tau,
\]
\[
u = U_\sigma(T, 0)u_T, \ \forall \ T > 0.
\] (16)
Next, we want to prove
\[
\liminf_{n \rightarrow +\infty} \|U_\sigma^n(t_n, \tau)u^n_T\| = \liminf_{n \rightarrow +\infty} \|U_\sigma^n(T, 0)u^n_T\| \geq \|\nu\|,
\] (17)
and
\[
U_\sigma^n(t_n, \tau)u^n_T \rightarrow \nu \text{ in } H \text{ as } n \rightarrow +\infty,
\] (18)
however, Theorem 2.1 ensures (18) holds. Next, we only need to prove (17). To this end, we shall prove
\[
\liminf_{n \rightarrow +\infty} \|U_\sigma^n(t_n, \tau)u^n_T\| = \liminf_{n \rightarrow +\infty} \|U_\sigma^n(T, 0)u^n_T\| \geq \|\nu\|,
\] (19)
\[
\limsup_{n \rightarrow +\infty} \|U_\sigma^n(t_n, \tau)u^n_T\| = \limsup_{n \rightarrow +\infty} \|U_\sigma^n(T, 0)u^n_T\| \leq \|\nu\|.
\] (20)
However, the weak convergence of corresponding sequences ensures that (19) is true, so our aim is only to prove (20).
By the weak convergence and norm convergence
\[
\limsup_{n, T \rightarrow \infty} \|U_\sigma^n(T, 0)u^n_T\|^2 = \lim_{n \rightarrow +\infty} \|U_\sigma^n(t_n, \tau)u^n_T\|^2 = \|U_\sigma(t, \tau)u_T\|^2 = \|\nu\|^2.
\] (21)
to arrive
\[
\lim_{n \rightarrow +\infty} \|U_\sigma^n(t_n, \tau)u^n_T - \nu\| = 0.
\] (22)

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References


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