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Remark on Uniform Attractor for the 3D Non-autonomous Brinkman-Forchheimer Equation

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Abstract

In this paper, we show the existence result of uniform attractors for the 3D non-autonomous Brinkman-Forchheimer equation equation.

Keywords: processes, upper semicontinuity, uniform attractor

1 Introduction

Let $\Omega \in R^3$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. We consider the large time behavior for a non-autonomous 3D Brinkman-Forchheimer equation:

$$u_t - \nu\triangle u + \alpha u + \beta|u|u + \gamma|u|^2u + \nabla p = \sigma(t,x), \tag{1}$$

$$divu = 0, \quad x \in \Omega, \quad t \in [\tau, +\infty), \tag{2}$$

$$u(t,x)|_{\partial\Omega} = 0, \ t \in [\tau, +\infty), \tag{3}$$

$$u(\tau,x) = u_\tau(x), \ x \in \Omega, \ \tau \geq 0. \tag{4}$$

Here $(x,t) \in \Omega \times [\tau, +\infty)$, $u = u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$ is the velocity vector field, p is the pressure, $\nu > 0$ is the Brinkman kinematic viscosity

coefficient, $\alpha > 0$ is the Darcy coefficient, $\beta > 0$ and $\gamma > 0$ are the Forchheimer coefficients.

Let us recall some known results for the Brinkman-Forchheimer equation. [3] and [10] investigated the physics significance and theoretical development of the 3D Brinkman-Forchheimer equation respectively, when the Reynold number for fluid is low, [3] shows the Darcy's Law also holds except some corrections. For the autonomous Brinkman-Forchheimer equation, it was shown in [1], [2], [4] and [6] that the Brinkman-Forchheimer equation has global solutions and the solution is continuous dependence on the coefficients, moreover, they also derived the convergence of corresponding solutions as coefficients tend to zero. [8], [5] and [9] obtained the existence of global attractor of Brinkman-Forchheimer equation by different methods. However, there are fewer results for the non-autonomous case, [7] deduced the existence of $\mathcal{D}$-pullback attractors for 3D non-autonomous Brinkman-Forchheimer equation by establishing the $\mathcal{D}$-pullback asymptotical compactness of θ-cocycle recently, [11] gave the existence of uniform attractor of 3D non-autonomous Brinkman-Forchheimer equation by continuous method.

In this paper, we shall describe our main result and some simple proof of [11].

2 Main Results

The family of functions $L^2_{loc}(R;H)$ denote a local Bochner integration function class, $L^2_b(R;H)$ denotes all translation bounded functions which satisfies $\sup\limits_{t\in R}\int_t^{t+1}\|\sigma(s,x)\|^2_H ds < +\infty$ for all $\sigma \in L^2_{loc}(R;H)$, i.e., σ is translation bounded in $L^2_{loc}(R;H)$. Obviously, $L^2_b(R;H) \subset L^2_{loc}(R;H)$.

The problem (1)-(2) can be written as an abstract form

$$u_t + \nu Au + \alpha u + B(u) = \sigma(t,x), \tag{5}$$

$$divu = 0, \tag{6}$$

where the pressure p has disappeared by force of the application of the Leray-Helmholtz projection P, $B(u) = PF(u)$, $F(u) = \beta|u|u + \gamma|u|^2u$. Clearly, system (5)-(6) is equivalent to (1)-(2).

The existence and uniqueness of global solution for (1)-(4) can be derived by standard method as in [9], [2] or [4]:

Theorem 2.1 *Assume $\sigma \in L^2_{loc}(R,H)$, $u_\tau \in H$, then problem (1)-(4) possesses a unique global solution $u(t,x)$ which satisfies*

$$u \in C([\tau,+\infty);H) \cap L^2(\tau,T;V) \cap L^4(\tau,T;(L^4(\Omega))^3). \tag{7}$$

Moreover, we choose a non-autonomous external force $\sigma_0(t,x) \in L^2_b(R,H)$ *and fixed, the global solution* $u(t,x)$ *generates a process* $\{U_\sigma(\tau,t)\}(\tau \in R,\ t > \tau,\ \sigma \in \Sigma)$ *which is continuous with respect to* u_τ*, where* σ *is a symbol which belong to the symbol space* $\Sigma = \mathcal{H}(\sigma_0) = \Big[\{\sigma_0(s+h)|h \in R\}\Big]_{L^2_{loc}(R,H)}$, $[\cdot]_E$ *means the closure in the topology* E*.*

Theorem 2.2 *Assume that* $u_\tau \in H$, $\sigma \in \Sigma \subset L^2_{loc}(\tau,+\infty;H)$*, then the family of processes* $\{U_\sigma(t,\tau), t \geq \tau \in R\}, (\sigma \in \mathcal{H}(\sigma_0))$ *generated by the global solution of problem (1)-(4) possesses a uniform (w.r.t.* $\sigma \in \Sigma = \mathcal{H}(\sigma_0)$*) attractor* $\mathcal{A}_{\mathcal{H}(\sigma_0)} = \mathcal{A}_\Sigma$ *in* H*.*

3 Discussion

Choosing an arbitrary non-autonomous force $\sigma_0(x,t) \in L^2_b(R,H)$ and then fixed, i.e.,

$$\sup_{t\in R}\int_t^{t+1} \|\sigma_0(s)\|^2_H ds < +\infty$$

taking $\Sigma = \mathcal{H}(\sigma_0)$ (Defined in Theorem 2.1) as the symbol space of problem (1)-(4), $\forall\ \sigma \in \Sigma$ is called the symbol of the system (1)-(4). Obviously, $\mathcal{H}(\sigma_0)$ is strictly invariant under the acting of the translation semigroup $\{S(h)\}_{h\geq 0}$, i.e., $S(h)\mathcal{H}(\sigma_0) \equiv \mathcal{H}(\sigma_0)$ for all $h \geq 0$.

From Theorem 2.1, the global solution generates processes class $\{U_\sigma(t,\tau), t \geq \tau, \tau \in R\}, \sigma \in \mathcal{H}(\sigma_0)$, i.e., $U_\sigma(t,\tau)u_\tau = u(t)$, where $u(t)$ is the solution of problem (1)-(4) with symbol $\sigma \in \Sigma$ and initial data $u_\tau \in H$.

Lemma 3.1 *Let the external force* $\sigma \in \Sigma$, $u_\tau \in H$*, then the process has a bounded uniform (w.r.t.* $\sigma \in \mathcal{H}(\sigma_0)$*) absorbing set* B_0 *in* H*, where* $B_0 = \Big\{u \in H : \|u\|_H \leq C\|\sigma_0\|_{L^2_b(R;H(\Omega))} \doteq \rho\Big\}$ *is a bounded set in* H*.*

Proof. Multiplying (1) with u and integrating on Ω, by the Young inequality we conclude

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\|\nabla u\|^2 + \alpha\|u\|^2 + \beta\|u\|^3_{L^3} + \gamma\|u\|^4_{L^4} &= \int_\Omega \sigma(t,x)u dx \\ &\leq \frac{\alpha}{2}\|u\|^2 + \frac{2\|\sigma\|^2}{\alpha}, \quad (8)\end{aligned}$$

then integrating over $[\tau,t]$, it follows

$$\begin{aligned}&\|u\|^2 + \int_\tau^t \Big(2\nu\|\nabla u\|^2 + \alpha\|u\|^2 + 2\beta\|u\|^3_{L^3} + 2\gamma\|u\|^4_{L^4}\Big)ds \\ \leq\ & \frac{4}{\alpha}\int_\tau^t \|\sigma(s)\|^2 ds + \|u_\tau\|^2, \quad (9)\end{aligned}$$

hence,

$$\|u\|^2 + \int_\tau^t \Big(2\lambda_1\nu + \alpha\Big)\|u\|^2 ds \le \frac{4}{\alpha}\int_\tau^t \|\sigma(s)\|^2 ds + \|u_\tau\|^2, \tag{10}$$

where λ_1 is the first eigenvalue in the Poincaré inequality.

By Gronwall's inequality, we derive

$$\begin{aligned}
\|u(t)\|^2 &\le \|u_\tau\|^2 e^{-(2\lambda_1\nu+\alpha)(t-\tau)} + \frac{4}{\alpha}\int_\tau^t e^{(2\lambda_1\nu+\alpha)(s-t)}\|\sigma(s)\|^2 ds \\
&\le \|u_\tau\|^2 e^{-(2\lambda_1\nu+\alpha)(t-\tau)} + \frac{4}{\alpha}\Big(\int_{t-1}^t e^{(2\lambda_1\nu+\alpha)(s-t)}\|\sigma(s)\|^2 ds \\
&\quad + \int_{t-2}^{t-1} e^{(2\lambda_1\nu+\alpha)(s-t)}\|\sigma(s)\|^2 ds + \cdots\Big) \\
&\le \|u_\tau\|^2 e^{-(2\lambda_1\nu+\alpha)(t-\tau)} + \frac{4}{\alpha}\Big(\int_{t-1}^t \|\sigma(s)\|^2 ds \\
&\quad + e^{-(2\lambda_1\nu+\alpha)}\int_{t-2}^{t-1}\|\sigma(s)\|^2 ds + e^{-2(2\lambda_1\nu+\alpha)}\int_{t-3}^{t-2}\|\sigma(s)\|^2 ds + \cdots\Big) \\
&\le \|u_\tau\|^2 e^{-(2\lambda_1\nu+\alpha)(t-\tau)} + \frac{4}{\alpha}\Big(1 + \frac{1}{(2\lambda_1\nu+\alpha)}\Big)\|\sigma(s)\|^2_{L_b^2(R;H)} \\
&\le \|u_\tau\|^2 e^{-(2\lambda_1\nu+\alpha)(t-\tau)} + \frac{4}{\alpha}\Big(1 + \frac{1}{(2\lambda_1\nu+\alpha)}\Big)\|\sigma_0\|^2_{L_b^2(R;H)},
\end{aligned} \tag{11}$$

choosing $\|u_\tau\|^2 e^{-(2\lambda_1\nu+\alpha)(t-\tau)} \le \frac{4}{\alpha}\Big(1 + \frac{1}{(2\lambda_1\nu+\alpha)}\Big)\|\sigma_0\|^2_{L_b^2(R;H)}$, then there exists a time $t_0 = t_0(\alpha, \lambda, \|\sigma_0\|^2_{L_b^2(R;H)})$ such that $B_0 = \Big\{u : \|u\|^2 \le \rho^2\Big\}$, where $\rho^2 = \frac{8}{\alpha}\Big(1 + \frac{1}{(2\lambda_1\nu+\alpha)}\Big)\|\sigma_0\|^2_{L_b^2(R;H)}$, i.e., B_0 is the uniformly (w.r.t. $\sigma \in \Sigma$) absorbing ball for the process $\{U_\sigma(t,\tau)\}$ in H.

Lemma 3.2 *For any $\sigma \in \mathcal{H}(\sigma_0)$, $u_\tau \in H$, the family of processes $\{U_\sigma(t,\tau), t \ge \tau \in R\}, \sigma \in \mathcal{H}(\sigma_0)$ defined on H, corresponding to equations (1)-(4) is uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact in H.*

Proof. Assume that $\{u_\tau^n\}$ is a bounded sequence in H, $\{\sigma^n\} \subset \mathcal{H}(\sigma_0)$ and $\{t_n\} \in (\tau, +\infty)$, $t_n \to +\infty$ as $n \to +\infty$.

From the proof of the existence of uniformly absorbing set, we see that for any fixed $\tau \in R$, there exists a time $T_0 = T_0(\rho, \tau)$ dependent on radius ρ of the absorbing ball and τ, such that for all $t_n \ge T_0$, $\{U_{\sigma^n}(t_n, \tau)u_\tau^n\} \subseteq B_0$, B_0 is defined in Lemma 3.1.

From Theorem 2.1, the sequence $\{U_{\sigma^n}(t_n, \tau)u_\tau^n\}$ is weakly precompact in H and we have

$$U_{\sigma^n}(t_n, \tau)u_\tau^n \rightharpoonup u \text{ weakly in } H \text{ as } n \to +\infty \tag{12}$$

for some $u \in H$ and some subsequence (still denoted by) $U_{\sigma^n}(t_n, \tau)u_\tau^n$.

Similarly, for each $T > 0$ and $t_n \geq T_0 + T$,

$$u_T^n \doteq U_{\sigma^n}(t_n - T, \tau)u_\tau^n \rightharpoonup u_T \ weakly \ in \ H \ as \ n \to +\infty \tag{13}$$

for some $u_T \in H$.

Noting that the translation semigroup $\{S(h) : h \geq 0\}$ satisfies

$$U_{S(h)\sigma}(t, \tau) = U_\sigma(t + h, \tau + h), \ \ \forall \ h \geq 0, \ t \geq \tau \in R, \ \forall \ \sigma \in \mathcal{H}(\sigma_0), \tag{14}$$

we see that

$$\begin{aligned} U_{\sigma^n}(t_n, \tau) &= U_{\sigma^n}(t_n, t_n - T)U_{\sigma^n}(t_n - T, \tau) \\ &= U_{S(t_n - T)\sigma^n}(T, 0)U_{\sigma^n}(t_n - T, \tau), \ t_n - T \geq \tau. \end{aligned} \tag{15}$$

Let $\sigma_T^n = S(t_n - T)\sigma^n$, by (13)–(15), we have

$$\begin{aligned} &U_{\sigma^n}(t_n, \tau)u_\tau^n = U_{\sigma_T^n}(T, 0)U_{\sigma^n}(t_n - T, \tau)u_\tau^n = U_{\sigma_T^n}(T, 0)u_T^n, \ t_n - T \geq \tau, \\ &u = U_{\sigma_T}(T, 0)u_T, \ \forall \ T > 0. \end{aligned} \tag{16}$$

Next, we want to prove

$$\|U_{\sigma_T^n}(t_n, \tau)u_\tau^n\| \to \|u\| \ as \ n \to +\infty, \tag{17}$$

and

$$U_{\sigma_T^n}(t_n, \tau)u_\tau^n \rightharpoonup u \ in \ H \ as \ n \to +\infty, \tag{18}$$

however, Theorem 2.1 ensures (18) holds. Next, we only need to prove (17). To this end, we shall prove

$$\liminf_{n \to +\infty} \|U_{\sigma^n}(t_n, \tau)u_\tau^n\| = \liminf_{n \to +\infty} \|U_{\sigma_T^n}(T, 0)u_T^n\| \geq \|u\|, \tag{19}$$

$$\limsup_{n \to +\infty} \|U_{\sigma^n}(t_n, \tau)u_\tau^n\| = \limsup_{n \to +\infty} \|U_{\sigma_T^n}(T, 0)u_T^n\| \leq \|u\|. \tag{20}$$

However, the weak convergence of corresponding sequences ensures that (19) is true, so our aim is only to prove (20).

By the weak convergence and norm convergence

$$\limsup_{n,T \to \infty} \|U_{\sigma_T^n}(T, 0)u_T^n\|^2 = \lim_{n \to \infty} \|U_{\sigma^n}(t_n, \tau)u_\tau^n\|^2 = \|U_\sigma(t, \tau)u_\tau\|^2 = \|u\|^2. \tag{21}$$

to arrive

$$\lim_{n \to \infty} \|U_{\sigma_T^n}(t_n, \tau)u_\tau^n - u\| = 0. \tag{22}$$

For more details, see Yang, Qin and Zhang's paper [11] in the year 2011.

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