Fractal Properties of Functions
Defined in Terms of Q-Representation

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Abstract. In this paper we study differential, self-affine, fractal and integral properties of a generalization of the Sierpinski function whose argument is represented in the form of $Q_s$-expansion and value represented in the form of $G_3$-expansion.

Keywords: continuous nowhere differentiable function, nowhere monotonic function, levels sets of function, self-affine properties function, $Q$-representation of real numbers

1. Introduction

Today, continuous nowhere differentiable functions more often attract attention of researchers and appear in both theoretical and applied studies [1], [2], [6], [9]. The theory of fractals opens up special conditions for their theoretical analysis. The Banach-Mazurkiewicz theorem stimulates additional philosophical and methodological interest in such functions. It states that almost all in the sense of Baire category continuous functions on $[0,1]$ are nowhere differentiable.
One of the simplest examples of a continuous non-differentiable function is the W. Sierpiński function [1] defined in terms of ternary and quinary representations of number. Q-representation is a generalization of s-adic numeration system [5]. Formal generalization of the Sierpiński function using $Q_s$- and $Q_3$-representation provides continuum family of functions depending on a finite number of parameters. Constructed family of functions has complicated differential, integral and fractal properties. We understand fractal properties as dimensional properties of graphs considered as sets in the space $R_2$ and level sets of function.

This paper is devoted to specified family of generalized Sierpiński functions. We study properties of functions related to self-affinity of graph.

2. Q-REPRESENTATION OF REAL NUMBERS

Let $1 < s$ be a fixed positive integer, and let $Q = \{q_0, q_1, \ldots, q_{s-1}\}$ be a fixed set with the following properties

1) $q_i > 0, \ i \in A_s = \{0, 1, \ldots, s - 1\}$,

2) $\sum_{i=0}^{s-1} q_i = 1$.

It is known (see, e.g., [5]) that for any $x \in [0, 1]$ there exists a sequence $(\alpha_k)$, $\alpha_k \in A_s$ such that

$$x = \beta_{\alpha_1} + \sum_{i=2}^{\infty} \left( \beta_{\alpha_i} \prod_{j=1}^{i-1} q_{\alpha_j} \right) \equiv \Delta^Q_{\alpha_1 \alpha_2 \ldots \alpha_k \ldots}$$

where $\beta_0 = 0, \beta_i = \sum_{j=0}^{i-1} q_j, i \in A_s \setminus \{0\}$.

The representation of the real number $x$ in the form (1) is said to be the Q-expansion and its symbolic notation $\Delta^Q_{\alpha_1 \alpha_2 \ldots \alpha_k \ldots}$ is said to be the Q-representation of $x$. The number $\alpha_k = \alpha_k(x)$ is said to be the k-th Q symbol of $x$. If $q_0 = q_1 = \ldots = q_{s-1} = \frac{1}{s}$, then Q-representation is the usual s-adic expansion. We note that Q-symbol of number $x$ is the index in the expansion (1). Sometimes in order to emphasize that Q-symbol belongs to the alphabet of s-adic numeration system we use notation $Q_s$.

Any number $x$ has no more than two formally different Q-representations. These numbers have the form $\Delta^Q_{\alpha_1 \alpha_2 \ldots \alpha_k \ldots} = \Delta^Q_{\alpha_1 \alpha_2 \ldots \alpha_{k-1} \alpha_k(0)} = \Delta^Q_{\alpha_1 \alpha_2 \ldots \alpha_{k-1} \alpha_k(1)}$ and they are called Q-rational. All other numbers are Q-irrational. Any Q-rational number has two different Q-representations, and any Q-irrational number has a unique Q-representation.

Let $(c_1, \ldots, c_m)$ be a fixed set of symbols from the alphabet $A_s$. Cylinder of rank $m$ with the base $c_1 \ldots c_m$ is a set of numbers $x \in [0, 1]$ with Q-representation $\Delta^Q_{c_1 c_2 \ldots c_m}$ such that $\alpha_j(x) = c_j, j = \overline{1, m}$.
Lemma 2.1. [3] The cylinder is a closed interval with endpoints

\[ a = \beta_{c_1} + \sum_{k=2}^{m} (\beta_{c_1} \prod_{j=1}^{k-1} q_{c_j}), \quad b = a + \prod_{i=1}^{m} q_{c_i}. \]

Properties of cylinder:

1) \( |\Delta_{c_1 \ldots c_m}^Q| = \prod_{i=1}^{s-1} q_{c_i}, \)
2) \( \Delta_{c_1 \ldots c_m}^Q = \bigcup_{i=0}^{s} \Delta_{c_1 \ldots c_m}^Q, \)
3) \( \max \Delta_{c_1 \ldots c_m}^Q = \min \Delta_{c_1 \ldots c_m}^Q, \quad i = 0, s - 2, \)
4) \( \bigcap_{n=1}^{\infty} \Delta_{c_1 \ldots c_m}^Q = x \equiv \Delta_{c_1 \ldots c_m}. \)

The properties 1-4 show that geometry of Q-representation is self-similar and has zero redundancy.

3. Definition of function

Let \( 3 < s \) be a fixed odd positive integer, \( A_s = \{0, 1, \ldots, s - 1\} \) be an alphabet, and let \( Q_s \) and \( G_3 \) be two given Q-representations.

We will define a discrete function on \( A_s \) as follows

(2) \[ \gamma(\alpha) = \begin{cases} 0 & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha \in A_s \setminus \{0, s - 1\} \\ 2 & \text{if } \alpha = s - 1. \end{cases} \]

For any sequence \( (\alpha_k) \in L = A_s^\infty = A_s \times A_s \times \ldots \), define sequence \( (c_k) \) as follows

(3) \[ c_1 = 0, \quad c_k = \begin{cases} c_{k-1} & \text{if } \alpha_{k-1} \in A_s \setminus \{2, 4, \ldots, s - 3\}, \\ 1 - c_{k-1} & \text{if } \alpha_{k-1} \in \{2, 4, \ldots, s - 3\}. \end{cases} \]

Let the argument of function is in the form of \( Q_s \)-representation

(4) \[ x = \varphi_{\alpha_1} + \sum_{i=2}^{\infty} \left( \varphi_{\alpha_1} \prod_{j=1}^{i-1} q_{\alpha_j} \right) \equiv \Delta_{\alpha_1 \alpha_2 \ldots \alpha_k}^{Q_s} \quad \alpha_k \in A_s, \]

where \( Q_s = \{q_0, q_1, \ldots, q_{s-1}\}, \quad \varphi_0 = 0, \quad \varphi_i = \sum_{j=0}^{i-1} q_j, \quad i \in A_s \setminus \{0\}, \)

and value of function has the following \( G_3 \)-representation

(5) \[ f(x) = \Delta_{\beta_1 \beta_2 \ldots}^{G_3} \equiv \psi_{\beta_1} + \sum_{i=2}^{\infty} \left( \psi_{\beta_1} \prod_{j=1}^{i-1} q_{\beta_j} \right), \]
\[ G_3 = \{g_0, g_1, g_2\}, \beta_k \in \{0, 1, 2\} = A_3, \psi_0 = 0, \psi_j = \sum_{i=0}^{j-1} g_i, j \in A_3 \setminus \{0\} \] and

(6) \[ \beta_1 = \gamma(\alpha_1), \quad \beta_k = \begin{cases} \gamma(\alpha_k), & \text{if } c_k = 0, \\ 2 - \gamma(\alpha_k), & \text{if } c_k \neq 0. \end{cases} \]

To prove some statements we need another form for \( \beta_k \):

(7) \[ \beta_k = \begin{cases} 0 & \text{if } \alpha_k = 0 \text{ and } c_k = 0, \\ 1 & \text{if } \alpha_k \in A_3 \setminus \{0, s - 1\}, \\ 2 & \text{if } \alpha_k = s - 1 \text{ and } c_k = 0. \end{cases} \]

**Remark 3.1.** It is easy to see that \( \beta_k \) depends on \((\alpha_1, \alpha_2, \ldots, \alpha_k - 1)\) but it depend only on \( \alpha_k \) if all \( \alpha_i \in A_3 \setminus \{0, s - 1\} \) (\( i = 1, k - 1 \)).

**4. Continuity of function \( f \)**

There are the following two different \( Q_s \)-representation for each rational point \( x \): \( x \equiv \Delta^{Q_s}_{\alpha_1 \alpha_2 \cdots \alpha_{k-1} \alpha_k(0)} = \Delta^{Q_s}_{\alpha_1 \alpha_2 \cdots \alpha_{k-1} \alpha_k(1)} \equiv x^* \). Therefore, the function \( f \) is well defined if for above two \( Q_s \)-expansions the correspondence is uniquely determined. We show that \( f(x) = f(x^*) \). Consider the difference

\[
\begin{align*}
\Delta f(x) &= \prod_{i=1}^{k-1} g_{\beta_i}(\psi_{\beta_k} - \psi_{\beta_k^*}) + \prod_{i=1}^{k-1} g_{\beta_i} \left( \sum_{n=k+1}^{\infty} (\psi_{\beta_n} \prod_{j=k}^{n-1} g_{\beta_j}) - \sum_{n=k+1}^{\infty} (\psi_{\beta_n^*} \prod_{j=k}^{n-1} g_{\beta_j}) \right) \\
&= \prod_{i=1}^{k-1} g_{\beta_i} \left( \psi_{\beta_k} - \psi_{\beta_k^*} \right) - \prod_{i=1}^{k-1} g_{\beta_i} \left( \sum_{n=k+1}^{\infty} \psi_{\beta_n} \prod_{j=k}^{n-1} g_{\beta_j} \right) \\
&= \prod_{i=1}^{k} g_{\beta_i} - \prod_{i=1}^{k-1} g_{\beta_i} \left( \psi_{\beta_k} - \psi_{\beta_k^*} \right) = 0.
\end{align*}
\]

Similarly, we can show that \( f(x) = f(x^*) \) for \( c_k = 1 \).

2) if \( c_{k+1}(x) \neq c_k = c_{k+1}(x^*) \) (or \( c_{k+1}(x) = c_k \neq c_{k+1}(x^*) \)), then \( \alpha_k(x) \in \{2, 4, \ldots, s - 3\} \) or \( \alpha_k(x^*) \in \{2, 4, \ldots, s - 3\} \). Therefore, using (7) we have \( \beta_n = \beta_n^* \) (\( n = k + i \), \( i \in \mathbb{N}_0 \)). So \( f(x) = f(x^*) = 0 \).

**Theorem 4.1.** The function \( f \) is continuous.

**Proof.** Let \( x_0 \) be an arbitrary point belonging to the interval \([0, 1]\). We show that \( \lim_{\Delta x \to x_0} |f(x) - f(x_0)| = 0 \).
First consider the case when the \( x_0 \) is a \( Q_s \)-irrational point. For any \( x \in [0, 1] \) there exists \( m = m(x) \) such that

\[
\begin{cases}
\alpha_i(x) = \alpha_i(x_0), & i = 1, m - 1, \\
\alpha_m(x) \neq \alpha_m(x_0),
\end{cases}
\]

and the condition \( x \to x_0 \) is equivalent to \( m \to \infty \). So,

\[
|f(x) - f(x_0)| = \left| \sum_{i=m}^{\infty} \psi_{\beta_i} \prod_{j=1}^{i-1} g_{\beta_j} - \sum_{i=m}^{\infty} \psi_{\beta_i} \prod_{j=1}^{i-1} g_{\beta_j} \right| \leq \prod_{i=1}^{m-1} g_{\beta_i} \left| \frac{g_0 + g_1}{1 - g_2} \right| = \prod_{i=1}^{m-1} g_{\beta_i} \leq (\max \{g_0, g_1, g_2\})^{m-1} \to 0 \text{ by } m \to \infty.
\]

Hence the function is continuous in the \( Q_s \)-irrational points.

Now let \( x_0 \) be a \( Q_s \)-rational number, i.e., \( x_0 = \Delta_{\alpha_1 \alpha_2 \ldots \alpha_k}^Q(0) \). To prove the left continuity of function we use the representation \( \Delta_{\alpha_1 \alpha_2 \ldots \alpha_k}^{Q_s} = \alpha_1 \alpha_2 \ldots \alpha_{k-1}[\alpha_k - 1](s-1) \), and to prove the right continuity we use representation \( \Delta_{\alpha_1 \alpha_2 \ldots \alpha_k}^{Q_s} = \alpha_1 \alpha_2 \ldots \alpha_k(0) \). Repeating the same steps as for the \( Q_s \)-irrational point will complete the proof.

5. NOWHERE MONOTONIC FUNCTIONS AND PROPERTIES OF LEVEL SETS.

**Definition 5.1.** The interval \( \left( \Delta_{\alpha_1 \alpha_2 \ldots \alpha_m}^{Q_s}(0), \Delta_{\alpha_1 \alpha_2 \ldots \alpha_m}^{Q_s}(s-1) \right) \) is called the cylindrical interval of rank \( m \) with the base \( \alpha_1 \alpha_2 \ldots \alpha_m \). Corresponding increment of the function on this interval is defined by equality

\[
\mu_f(\Delta_{\alpha_1 \alpha_2 \ldots \alpha_m}^{Q_s}) = f(\Delta_{\alpha_1 \alpha_2 \ldots \alpha_m}(s-1)) - f(\Delta_{\alpha_1 \alpha_2 \ldots \alpha_m}(0))
\]

**Definition 5.2.** Function \( f \) is called nowhere monotonic on the interval \( [0, 1] \) if it is continuous on this interval and has no intervals of monotonicity.

**Lemma 5.3.** Function \( f \) is nowhere monotonic and the following equality holds: \( \mu_f(\Delta_{\alpha_1 \alpha_2 \ldots \alpha_m}) = (-1)^{m-1} \prod_{i=1}^{m} g_{\beta_i} \).

**Proof.** To prove nowhere monotonicity of function \( f \) it is sufficient to show that for any cylinder \( \Delta_{\alpha_1 \alpha_2 \ldots \alpha_m}^{Q_s} \) of rank \( m \) there exists cylinder \( \Delta_{\alpha_1 \alpha_2 \ldots \alpha_m \alpha_{m+1}}^{Q_s} \) (\( j \in A_s \)) of rank \( m + 1 \) such that the increments \( \mu_f(\Delta_{\alpha_1 \alpha_2 \ldots \alpha_m}^{Q_s}) \) and \( \mu_f(\Delta_{\alpha_1 \alpha_2 \ldots \alpha_{m+1}}^{Q_s}) \) have different signs. From conditions (7) we have

1) for \( c_m = 0 \),

\[
\mu_f(\Delta_{\alpha_1 \alpha_2 \ldots \alpha_m}^{Q_s}) = \prod_{i=1}^{m} g_{\beta_i} \left( \psi_2 + \sum_{j=1}^{\infty} \psi_2 \prod_{k=m+1}^{j-1} g_2 \right) = \prod_{i=1}^{m} g_{\beta_i},
\]

\[
\mu_f(\Delta_{\alpha_1 \alpha_2 \ldots \alpha_{m+1}}^{Q_s}) = - \prod_{i=1}^{m+1} g_{\beta_i} \left( \psi_2 + \sum_{j=1}^{\infty} \psi_2 \prod_{k=m+1}^{j-1} g_2 \right) = - \prod_{i=1}^{m+1} g_{\beta_i},
\]

2) for \( c_m = 1 \),
\[ \mu_f(\Delta_{a_1a_2...a_m}) = - \sum_{i=1}^{m} g_{\beta_i} \left( \psi_2 + \sum_{j=1}^{\infty} \sum_{k=m+1}^{\infty} g_2 \right) = - \sum_{i=1}^{m} g_{\beta_i}. \]

\[ \mu_f(\Delta_{a_1a_2...a_m^2}) = \prod_{i=1}^{m+1} g_{\beta_i} \left( \psi_2 + \sum_{j=1}^{\infty} \sum_{k=m+1}^{\infty} g_2 \right) = \prod_{i=1}^{m+1} g_{\beta_i}. \]

**Definition 5.4.** The level set \( y_0 \) of function \( f \) is the set \[ f^{-1}(y_0) = \{ x : f(x) = y_0 \}. \]

**Lemma 5.5.** If \( y_0 = \Delta_{(1)}^{G_3} (m \in \mathbb{N}_0) \), then \( f^{-1}(y_0) = C[Q_s, V] \equiv \{ x : \alpha_i(x) \in V = A \setminus \{0, s - 1\} \} \) i.e., level set \( y_0 \) has the properties: 1) it is a continuum set; 2) it is a nowhere dense set; 3) it is a set of zero Lebesgue measure \( \lambda \); 4) its Hausdorff-Besicovitch dimension is the solution of equation \( q_1^x + q_2^x + \ldots + q_s^x = 1. \)

**Proof.** First we prove the lemma for \( m = 0 \), i.e., for \( y_0 = \Delta_{(1)}^{G_3} \).

It is clear that for any point \( x \in C[Q_s, V] \) we have \( f(x) = y_0 \), i.e., \( f^{-1}(y_0) \supset C[Q_s, V] \). Moreover, \( \beta_m(y_0) = 1 \) if and only if \( \alpha_m(f^{-1}(y_0)) \in A \setminus \{0, s - 1\} \), independently of \( n \). So, \( f^{-1}(y_0) = C[Q_s, V] \). It is known that \( C[Q_s, V] \) has properties mentioned in the lemma.

Now let \( m \neq 0 \), i.e., \( y_0 = \Delta_{d_1d_2...d_m(1)}^{G_3} \). Then

\[ H = \bigcup_{i=1}^{s-1} \bigcup_{i_2=0}^{s-1} \bigcup_{i_m=0}^{s-1} \left( \Delta_{d_1d_2...d_m}^{G_3} \cap f^{-1}(y_0) \right). \]

If the set \( \Delta_{d_1d_2...d_m}^{G_3} \cap f^{-1}(y_0) \) is not empty, then arbitrary \( x \) belonging to this set has the form \( x = \Delta_{d_1d_2...d_m}^{G_3} \cap V \) (\( j \in \mathbb{N} \)).

The set \( \Delta_{d_1d_2...d_m}^{G_3} \cap f^{-1}(y_0) \) can be empty or not empty. Moreover, it is not empty if \( m \) is an ordered set \((\beta_1, \beta_2, \ldots, \beta_m)\) such that

\[ \beta_1 = \beta_1(i_1) = d_1, \quad \beta_2 = \beta_2(i_1, i_2) = d_2, \quad \ldots, \quad \beta_m = \beta_m(i_1, i_2, \ldots, i_m) = d_m. \]

In this case the set \( \Delta_{d_1d_2...d_m}^{G_3} \cap f^{-1}(y_0) \) is \( \prod_{j=1}^{m} g_{i_j} \) times less “copy” than the set \( f^{-1}(\Delta_{d_1d_2...d_m}^{G_3}) \), since equality \( f^{-1}(\Delta_{d_1d_2...d_m}^{G_3}) = \sum_{j=1}^{\infty} \Delta_{d_1d_2...d_m}^{G_3} \cap f^{-1}(y_0) \) holds. Then \( f^{-1}(y_0) = C[Q_s, V_k] \), where \( V_k = \{ i_k \} \) \( (k = 1, m) \) \( V_{m+j} = V \).

Since minimal intervals containing sets from the union (8) are pairwise disjoint, the set \( f^{-1}(y_0) \) has properties mentioned in the lemma. \( \square \)

**Lemma 5.6.** If \( y_0 = \Delta_{\beta_1\beta_2...\beta_k...}^{G_3} \), where

\[ \left\{ \begin{array}{l}
\beta_m(y_0) = 1, \quad n \in \mathbb{N}, \\
\beta_j(y_0) \neq 1, \quad j \notin \{k_n\},
\end{array} \right. \]
then the set \( f^{-1}(y_0) \) is continuum and does not contain pair of points \( x_1 \) and \( x_2 \) such that

\[
\begin{align*}
\alpha_{k_n}(x_1) &= \alpha'_{k_n}(x_2) \\
\alpha_j(x_1) &\neq \alpha'_j(x_2), \ j \notin \{k_n\}.
\end{align*}
\]

(10)

**Proof.** Suppose that for the point \( y_0 = \Delta_{\beta_1\beta_2...\beta_k}^G \) conditions (9) and (10) hold and \( f^{-1}(y_0) = \{x_1, x_2\} \), where \( x_1 = \Delta_{\alpha_1\alpha_2...\alpha_k}^Q, \ x_2 = \Delta_{\alpha'_1\alpha'_2...\alpha'_k}^Q \) such that \( f(x_1) = y_0, \ f(x_2) = y_0 \). Then \( \exists k \in \mathbb{N} : \alpha_i = \alpha'_i \) for \( i = 1, k-1 \) and \( \alpha_k \neq \alpha'_k \). Then from (9) and (10) follows that \( \alpha_k, \alpha'_k \in \{0, s-1\} \), and hence \( \beta_k(f(x)) \neq \beta'_k(f(x')) \), i.e., \( f(x_1) - f(x_2) \neq 0 \). And this contradicts the condition of the lemma.

**Lemma 5.7.** If digits \( i_k \in A_3 \setminus \{1\} \) \( (k = 1, \infty) \) in the \( G_3 \)-representation of the point \( y_0 = \Delta_{i_1i_2...i_k}^G \), then the set \( f^{-1}(y_0) \) contains a single point.

**Proof.** Suppose that the set \( f^{-1}(y_0) \) contains at least two different points \( x = \Delta_{\alpha_1\alpha_2...\alpha_k}^Q, x' = \Delta_{\alpha'_1\alpha'_2...\alpha'_k}^Q \). Then there exists \( m \) such that \( \alpha_m \neq \alpha'_m \) but \( \alpha_i = \alpha'_i \) (for \( i < m \)).

From \( \alpha_i = \alpha'_i \) (for \( i < m \)) follows that \( \beta(\alpha_1, \alpha_2, \ldots, \alpha_{m-1}) = \beta(\alpha'_1, \alpha'_2, \ldots, \alpha'_{m-1}) \). Since \( \beta_m(f(x)) = i_m(y_0) = \beta_m(f(x')) \) and \( \alpha_m \neq \alpha'_m \) from (7) follows that \( c_{m-1}(x) \neq c_{m-1}(x') \), \( i_m(y_0) = 1 \). From conditions \( c_{m-1}(x) \neq c_{m-1}(x') \) follows that \( \exists\alpha_j (j < m) \) such that \( \alpha_j \neq \alpha'_j \), which contradicts the assumption of the lemma. While the condition \( i_m(y_0) = 1 \) contradicts the condition of the lemma. This contradiction proves the lemma.

**Lemma 5.8.** If \( G_3 \)-representation of the point \( y_0 = \Delta_{i_1i_2...i_k}^G \) contains exactly \( n \) digits "1", then the set \( f^{-1}(y_0) \) consists of the \( (s-2)^n \) points.

**Proof.** This statement is obvious. Since from (7) that in each place where \( \beta_i(y_0) = 1 \) are \( (s-2) \) alternatives, and all the other numbers remain fixed.

**Theorem 5.9.** Function \( f \) is non-differentiable in \( Q_s \)-rational point when conditions \( y_0 \geq \max\{q_0, q_{s-1}\} \) and \( g_2 \geq \max\{q_0, q_{s-1}\} \) hold.

**Proof.** To prove the statement it is sufficient to show that for each point \( x_0 \) there exists a sequence \( (x_m) \) converging to it such that the limit \( \lim_{m \to \infty} \frac{f(x_0) - f(x_m)}{x_0 - x_m} \) is infinite or does not exist.

Let \( x_0 \) is a \( Q_s \)-rational point, i.e.,

\[
x_0 = \Delta_{\alpha_1\alpha_2...\alpha_{k-1}\alpha_k}^Q = \Delta_{\alpha_1\alpha_2...\alpha_{k-1}[\alpha_k-1](s-1)}^Q = x_0'.
\]

We choose a sequence \( x_m \) such that

\[
x_m = \Delta_{\alpha_1\alpha_2...\alpha_{k-1}\alpha_k}^Q \underbrace{0...0}_{m}(s-1), \ x'_m = \Delta_{\alpha_1\alpha_2...\alpha_{k-1}[\alpha_k-1][s-1]...[s-1](0)}^Q
\]

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obviously, \( x_m \to x_0 + 0 \), \( x'_m \to x'_0 - 0 \). Then
\[
x_m - x_0 = q_0^m q_{\alpha_k} \prod_{i=1}^{k-1} q_{\alpha_i}, \quad x'_m - x'_0 = -q_0^m q_{[s-1]} q_{[\alpha_k-1]} \prod_{i=1}^{k-1} q_{\alpha_i}.
\]

If the derivative of the function \( f \) at the point \( x_0 \) exists, then
\[
f'(x_0) = \lim_{x_m \to x_0^+} \frac{f(x_m) - f(x_0)}{x_m - x_0} = \lim_{x_m \to x_0^-} \frac{f(x'_m) - f(x'_0)}{x'_m - x'_0}.
\]

So,
\[
\frac{f(x_m) - f(x_0)}{x_m - x_0} = \frac{\mu_f(\Delta^{Q_s}_{\alpha_1 \alpha_2 \ldots \alpha_k} q_0 \ldots q_0)}{x_m - x_0} = \begin{cases} 
\left( \frac{g_0}{q_0} \right)^m \frac{g_{\beta_k}}{q_{\alpha_k}} \prod_{i=1}^{k-1} g_{\beta_i} / q_{\alpha_i} & \text{if } c_k(x_0) = 0, \\
- \left( \frac{g_2}{q_0} \right)^m \frac{g_{\beta_k}}{q_{\alpha_k}} \prod_{i=1}^{k-1} g_{\beta_i} / q_{\alpha_i} & \text{if } c_k(x_0) \neq 0;
\end{cases}
\]

(11)

\[
\frac{f(x'_m) - f(x'_0)}{x'_m - x'_0} = \frac{-\mu_f(\Delta^{Q_s}_{\alpha_1 \alpha_2 \ldots \alpha_{k-1}} [s-1] \ldots [s-1])}{x'_m - x'_0} = \begin{cases} 
\left( \frac{g_2}{q_{[s-1]}} \right)^m \frac{g_{\beta'_k}}{q_{[\alpha_k-1]}} \prod_{i=1}^{k-1} g_{\beta_i} / q_{\alpha_i} & \text{if } c_k(x'_0) = 0, \\
- \left( \frac{g_0}{q_{[s-1]}} \right)^m \frac{g_{\beta'_k}}{q_{[\alpha_k-1]}} \prod_{i=1}^{k-1} g_{\beta_i} / q_{\alpha_i} & \text{if } c_k(x'_0) \neq 0,
\end{cases}
\]

(12)

where \( \beta'_k = \beta_k \) if \( \alpha_k \in A_s \setminus \{1, s-1\} \) and \( \beta'_k = |1 - \beta_k| \) if \( \alpha_k \in \{1, s-1\} \).

From the formulas (11) and (12) follows that function \( f \) is non-differentiable in \( Q_s \)-rational point, when conditions \( g_0 \geq \max\{q_0, q_{s-1}\} \) and \( g_2 \geq \max\{q_0, q_{s-1}\} \) hold.

**Theorem 5.10.** Function \( f \) is non-differentiable in \( Q_s \)-irrational point if \( \min g_i \geq \max q_j, \ j \in A_s, \ i \in A_3. \)

**Proof.** Let now \( x_0 \) be an arbitrary irrational number. Then for number \( x_0 \) there exists an infinite sequence \( (m_k) \) indexes such that \( \alpha_{m_k}(x_0) \in A_s \setminus \{2, 4, \ldots, s-3\} \). Consider the sequence \( (x_k) \) such that \( \alpha_i(x_k) = \alpha_i(x_0) \), when \( i \neq m_k \), and \( \alpha_{m_k}(x_k) = \left\lfloor \frac{\alpha_{m_k}(x_0)}{2} \right\rfloor \),

where \( \left\lfloor \frac{\alpha_{m_k}(x_0)}{2} \right\rfloor \) is an integer part of \( \frac{\alpha_{m_k}(x_0)}{2} \), when \( i = m_k \).
Obviously, the \( x_k \to x_0 \), when \( k \to \infty \). Then we have

\[
x_0 - x_k = \prod_{i=1}^{m_k-1} q_{a_i} \left( \varphi_{a_{m_k}(x_0)} - \varphi_{a_{m_k}(x_k)} \right) = \pm \prod_{i=1}^{m_k} q_{a_i}.
\]

\[
f(x_0) - f(x_k) = \prod_{i=1}^{m_k-1} g_{\beta_i} \left( \psi_{\beta_{m_k}(f(x_0))} - \psi_{\beta_{m_k}(f(x_k))} \right) = \pm \prod_{i=1}^{m_k} g_{\beta_i}.
\]

And hence

\[
\lim_{m \to \infty} \frac{f(x_0) - f(x_k)}{x_0 - x_k} = \pm \lim_{m \to \infty} \prod_{i=1}^{m_k} \frac{g_{\beta_i}}{q_{a_i}}.
\]

Therefore, if \( \min g_i \geq \max q_j \) (where \( j \in A_s \), \( i \in A_3 \)) the limit of the last relation is equal to \( \pm \infty \).

\[ \Box \]

6. Self-affine properties of function \( f \)

**Theorem 6.1.** Let \( x = \Delta_{Q_1 \alpha_1 \ldots \alpha_k} \), \( f(x) = \Delta_{G_1 \beta_1 \beta_2 \ldots \beta_k} \). Graph \( \Gamma_f = \{ (x', f(x')) \mid x' \in [0, 1] \} \) of the function \( f \) is self-affine set, and \( \Gamma_f = \bigcup_{i=0}^{s-1} \phi_i(\Gamma_f) \), \( \phi_i = \left\{ \begin{array}{ll} x' = \varphi_1 + q_i x' = \Delta_{Q_1 \alpha_1 \ldots \alpha_k}, & i \in A_s, \\
\phi_i(\Gamma_f) = c_2 \psi_2 + (1 - c_2) \psi_{\gamma(i)} + (-1)^{c_2} q_{\gamma(i)} f(x). & \end{array} \right. \)

**Proof.** Let \( G = \phi_1(\Gamma_f) \cup \phi_2(\Gamma_f) \cup \ldots \cup \phi_n(\Gamma_f) \), \( \phi_i(\Gamma_f) \neq \phi_j(\Gamma_f) \) for \( i \neq j \). We prove that \( \Gamma_f = G \).

To this end first we show that \( G \subset \Gamma_f \). Consider an arbitrary point \( M(x_M, y_M) \in G \), then there exists \( i \) such that \( x_M = q_i x + \varphi_i \), \( y_M = \psi_{\gamma(i)} + c_2 g_1 + (-1)^{c_2} q_{\gamma(i)} f(x) \).

It is easy to prove that \( f(x_M) = y_M \). So \( M \in \Gamma_f \).

We will now prove that \( \Gamma_f \subset G \), i.e., for any point \( M(x, y) \in \Gamma_f \), namely \( x = \Delta_{Q_1 \alpha_1 \ldots \alpha_k}, \ y = \Delta_{G_1 \beta_1 \beta_2 \ldots \beta_k} = c_2 \psi_2 + (1 - c_2) \psi_{\gamma(i)} + (-1)^{c_2} q_{\gamma(i)} f(x) \), there exists \( i \) such that \( M \in \phi_i(\Gamma_f) \). For this consider a point \( M^*(x^*, y^*) \in \Gamma_f \), i.e., \( x^* = \Delta_{Q_1 \alpha_1 \ldots \alpha_k}, \ y^* = \Delta_{G_1 \beta_1 \beta_2 \ldots \beta_k} \).

Since \( \phi_{\alpha_1}(M^*) = M \), we have \( M \in \phi_{\alpha_1}(\Gamma_f) \). Hence, \( \Gamma_f \equiv G \).

\[ \Box \]

**Theorem 6.2.** For the Lebesgue integral the following equality holds:

\[
\int_{0}^{1} f(x) dx = \frac{(g_0 + g_1)(1 - q_0) - g_1 \sum_{i=1}^{m+1} q_{2i-1}}{1 - q_0 g_0 - g_1 \left( \sum_{i=1}^{m+1} q_{2i-1} - \sum_{i=m}^{m} q_{2i} \right) - g_2 q_{s-1}}, \quad \text{where} \quad m = \frac{s - 3}{2}
\]

**Proof.** Using the additive property of the Lebesgue integral and self-affine properties of function we have
\[
\int_0^1 f(x)dx = \int_0^{\varphi_1} f(x)dx + \sum_{i=1}^{s-2} \int_{\varphi_i}^{\varphi_{i+1}} f(x)dx + \int_{\varphi_{s-1}}^{1} f(x)dx = \int_0^{\varphi_1} g_0 f(t)dt + \\
+ \sum_{i=1}^{m} \int_{\varphi_{2i-1}}^{\varphi_{2i+1}} (g_0 + g_1 f(t)) dt + \sum_{i=1}^{m} \int_{\varphi_{2i}}^{\varphi_{2i+1}} (g_0 + g_1 - g_2 f(t)) dt + \\
+ \int_{\varphi_s-1}^{1} (g_0 + g_1 + g_2 f(t)) dt = q_0 g_0 \int_0^{1} f(x)dx + g_0 \sum_{i=1}^{m+1} q_{2i-1} + \\
+ q_1 \sum_{i=1}^{m+1} q_{2i-1} \int_0^{1} f(x)dx + (g_0 + g_1) \sum_{i=1}^{m} q_{2i} - q_1 \sum_{i=1}^{m} q_{2i} \int_0^{1} f(x)dx + \\
+ (g_0 + g_1) q_{s-1} + g_2 q_{s-1} \int_0^{1} f(x)dx,
\]

\[
\left(1 - q_0 g_0 - g_1 \sum_{i=1}^{m+1} q_{2i-1} + g_1 \sum_{i=1}^{m} q_{2i} - g_2 q_{s-1}\right) \int_0^{1} f(x)dx = \\
q_0 \sum_{i=1}^{m+1} q_{2i-1} + (g_0 + g_1) \sum_{i=1}^{m} q_{2i} + (g_0 + g_1) q_{s-1},
\]

\[
\int_0^{1} f(x)dx = \frac{(g_0 + g_1) (1 - q_0) - q_1 \sum_{i=1}^{m+1} q_{2i-1}}{1 - q_0 g_0 - g_1 \left(\sum_{i=1}^{m+1} q_{2i-1} - \sum_{i=1}^{m} q_{2i}\right) - g_2 q_{s-1} + g_2 q_{s-1}},
\]

Theorem 6.3. Let \([0, 1] \ni u \) be a real number. Then for Lebesgue integral the following equality holds

\[
\int_0^u f(x)dx = \lim_{n \to \infty} \sum_{k=1}^{n} \left( S^{(k)} \prod_{i=1}^{k-1} q_{\alpha_i} \right), \text{ where}
\]

\[
S^{(k)} = \psi_{\alpha k} D_{k-1} + \prod_{n=1}^{k-1} g_{\beta_n} \sum_{j=0}^{\alpha-1} \left( q_j S^{(k)}_{\Delta_{\alpha_1 \alpha_2 \cdots \alpha_{k-1}}^k} \right),
\]

\[
D_{k-1} = \sum_{j=1}^{k-1} \left( \psi_{j} \prod_{n=1}^{j-1} g_{\beta_n} \right), \quad S^{(k)}_{\Delta_{\alpha_1 \alpha_2 \cdots \alpha_{k-1}}} = \psi_{\beta_k} + (-1)^{c_k} g_{\beta_k} I, \quad I = \int_0^{1} f(x)dx.
\]
Proof. Since $u \in [0, 1]$, it can be represented in the form

$$u = \varphi_{a_1} + \sum_{i=2}^{\infty} \left( \varphi_{a_i} \prod_{j=1}^{k-1} q_{a_j} \right), \quad f(u) = \psi_{\beta_1} + \sum_{i=2}^{\infty} \left( \psi_{\beta_i} \prod_{j=1}^{k-1} g_{\beta_j} \right).$$

We denote

$$\frac{u - \varphi_{a_1}}{q_{a_1}} = \xi_1 = \Delta_{a_2a_3\ldots a_k}\ldots, \quad \frac{u - \varphi_{a_1} - \varphi_{a_2}q_{a_1}}{q_{a_1}q_{a_2}} = \xi_2 = \Delta_{a_3a_4\ldots a_k}\ldots,$$

$$\frac{u - \varphi_{a_1} - \varphi_{a_2}q_{a_1} - \ldots - \varphi_{a_k} \prod_{i=1}^{k-1} q_{a_i}}{q_{a_1}q_{a_2}\ldots q_{a_k}} = \xi_k = \Delta_{a_{k+1}a_{k+2}\ldots}, \quad \xi_k \in [0, 1], \quad k = 1, 2, \ldots.$$

Using the additive property of the Lebesgue integral and self-affine properties

$$\int_0^u f(u) du = \int_0^{\xi_1} f(\xi_1) d\xi_1 + \int_{\varphi_{a_1}}^{\varphi_{a_1} + \varphi_{a_2}q_{a_1}} f(\xi_2) d\xi_2 + \ldots$$

of function we have

$$\varphi_{a_1} + \sum_{i=2}^{k-1} \left( \varphi_{a_i} \prod_{j=1}^{k-1} q_{a_j} \right) + \int_{\varphi_{a_1} + \sum_{i=2}^{k-1} \left( \varphi_{a_i} \prod_{j=1}^{k-1} q_{a_j} \right)}^{\varphi_{a_1} + \sum_{i=2}^{k-1} \left( \varphi_{a_i} \prod_{j=1}^{k-1} q_{a_j} \right)} f(\xi_k) d\xi_k + \ldots$$

Each summand of the formula determines a successive approximation of the value of the integral. Denote

by $S_{\Delta_{a_1}}^{(1)}$ the first approximation, were $i = 0, a_1 - 1$,

by $S_{\Delta_{a_1}}^{(2)}$ the second approximation, where $i = 0, a_2 - 1$,

by $S_{\Delta_{a_1a_2\ldots a_{k-1}}}^{(k)}$ the $k$-th approximation, where $i = 0, a_k - 1$. Now we have

$$S_{\Delta_{a_i}}^{(1)} = \psi_{\gamma(i)} + q_{\gamma(i)} I, \quad i \in A \setminus \{2, 4, \ldots, s - 3\},$$

$$S_{\Delta_{a_i}}^{(1)} = q_0 + q_1 - q_1 I, \quad j \in \{2, 4, \ldots, s - 3\},$$

$$S_{\Delta_{a_1a_2\ldots a_{k-1}}}^{(k)} = \psi_{\beta_k} + (-1)^{\gamma_{k+1}} g_{\beta_k} I, \quad \text{where}$$

$$I = \int_0^u f(x) dx, \quad c_{k+1}' = \begin{cases} c_k, & i \in A \setminus \{2, 4, \ldots, s - 3\}; \\ 1 - c_k, & i \in \{2, 4, \ldots, s - 3\}. \end{cases}$$

Hence, we have

$$\int_0^u f(x) dx = \sum_{i=0}^{\alpha_1 - 1} \int_{\varphi_i}^{\varphi_i + 1} \left( S_{\Delta_{a_i}}^{(1)} \right) d\xi_1 + q_{a_1} \sum_{i=0}^{\alpha_2 - 1} \int_{\varphi_i}^{\varphi_i + 1} \left( \psi_{\beta_i} + g_{\beta_i} S_{\Delta_{a_1a_2\ldots a_{k-1}}}^{(k)} \right) d\xi_2 + \ldots +$$

$$+ \prod_{i=1}^{k-1} q_{a_i} \sum_{i=0}^{\alpha_k - 1} \int_{\varphi_i}^{\varphi_i + 1} \left( \psi_{\beta_i} \prod_{j=1}^{k-1} g_{\beta_j} \right) + \prod_{n=1}^{k-1} g_{\beta_n} S_{\Delta_{a_1a_2\ldots a_{k-1}}}^{(k)} \right) d\xi_k + \ldots =$$
\[= \sum_{i=0}^{\alpha_1-1} \left( q_i S^{(1)}_i \right) + q_{\alpha_1} \sum_{i=0}^{\alpha_2-1} \left( q_i \left( \psi_\beta_1 + g_\beta_1 S^{(2)}_{\Delta_\alpha_1} \right) \right) + \]
\[+ q_{\alpha_1} q_{\alpha_2} \sum_{i=0}^{\alpha_3-1} \left( q_i \left( \psi_\beta_1 + \psi_\beta_2 g_\beta_1 + g_\beta_1 g_\beta_2 S^{(3)}_{\Delta_\alpha_1} \right) \right) + \ldots + \]
\[+ \prod_{i=1}^{k-1} q_{\alpha_i} \sum_{i=0}^{\alpha_k-1} \left( q_i \left( \sum_{j=1}^{k-1} \left( \psi_\beta_j \prod_{n=1}^{j-1} g_\beta_n \right) + \prod_{n=1}^{k-1} g_\beta_n S^{(k)}_{\Delta_\alpha_1 \alpha_2 \ldots \alpha_{k-1}} \right) \right) + \ldots \]

Denote
\[D_{k-1} = \sum_{j=1}^{k-1} \left( \psi_\beta_j \prod_{n=1}^{j-1} g_\beta_n \right), \quad S^{(k)} = \psi_\alpha k D_{k-1} + \prod_{n=1}^{k-1} g_\beta_n \sum_{i=0}^{\alpha_k-1} \left( q_i S^{(k)}_{\Delta_\alpha_1 \alpha_2 \ldots \alpha_{k-1}} \right), \]

then \[\int_0^u f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^n \left( S^{(k)} \prod_{i=1}^{k-1} q_{\alpha_i} \right). \quad \square \]

**Corollary 6.4.** If \( s = 5 \), \( Q_5 = \{ \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \} \), \( G_3 = \{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \} \) then \[\int_0^u f(x) \, dx = \frac{1}{2} \] and graph of the function \( f \) has the form

![Graph of the function](image)

**References**


Fractal properties of functions


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