On Some Fixed Point Theorems with \( \varphi \)-Contractions in Cone Metric Spaces Involving a Graph

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Abstract

In the present paper, we introduce \( \varphi \)-contractions defined on a cone metric space endowed with a graph without assuming the normality condition of cone. We establish fixed point results for such contractions which are extension of several known results. Also, an example have been given which satisfies our main result.

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1 Introduction

Since fixed point theory plays a significant role in mathematics and applied sciences such as optimization, mathematical models, economy and medicine. So, The metric fixed point theory has been researched extensively in the past two decades.
In 2007 Huang and Zhang [5] generalized the concept of metric spaces, replacing the set of real numbers by an ordered Banach space, hence they have defined the cone metric spaces. They also described the convergence of sequences and introduced the notion of completeness in cone metric spaces. They have proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality condition of a cone.

Jachymski [6] used the platform of graph theory instead of partial ordering in metric space. Also, a mapping on a complete metric space still has a fixed point as long as the mapping satisfies the contraction condition for pairs of points which from edges in the graph.

In this paper, motivated by the work of Huang and Zhang [5] and Jachymski [6], we introduced new contractions the mappings on complete cone metric space by using the concept of graph and obtained some fixed point theorems. Our results generalize and unify some results by the above mentioned authors.

2 Preliminaries

Let $E$ be a real Banach space and $K$ be a subset of $E$. $K$ is called a cone if and only if

i. $K$ is closed, nonempty and $K \neq \{0\}$,

ii. $a, b \in \mathbb{R}; a, b \geq 0; x, y \in K \Rightarrow ax + by \in K$,

iii. $x \in K$ and $-x \in K \Rightarrow x = 0$.

Given a cone $K \subset E$, we define a partial ordering $\leq$ with respect to $K$ by $x \leq y$ if and only if $y - x \in K$. We write $x < y$ if $x \leq y$ but $x \neq y$; $x \ll y$ if $y - x \in \text{int}K$, where $\text{int}K$ is the interior of $K$. Huang and Zhang [5] redefined cone metric spaces as:

**Definition 2.1** Let $X$ be nonempty set, $E$ be a real Banach space and $K \subset E$ be a cone. Suppose the mapping $d : X \times X \to E$ satisfies

$d1. \ 0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

$d2. \ d(x, y) = d(y, x)$ for all $x, y \in X$;

$d3. \ d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space. It is obvious that the concept of a cone metric space is more general than a metric space.

Let $\{x_n\}$ be a sequence in a cone metric space $X$ and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$
then \( x_n \) is called convergent sequence. If for every \( c \in E \) with \( 0 \ll c \) there is \( n_0 \in \mathbb{N} \) such that for all \( n, m > n_0 \), \( d(x_n, x_m) \ll c \) then \( x_n \) is called a Cauchy sequence in \( X \). A cone metric space \( X \) is said to be complete if every Cauchy sequence in \( X \) is convergent in \( X \).

It is known that \( \{x_n\} \) converges to \( x \in X \) if and only if \( d(x_n, x) \to 0 \) as \( n \to \infty \).

The following lemma has been given in [4] that we utilize them to prove our theorems.

**Lemma 2.2** Let \((X, d)\) be a cone metric space, \(u, v, w \in X\). Then

i. If \( u \ll v \) and \( v \ll w \), then \( u \ll w \).

ii. If \( u \leq v \) and \( v \ll w \), then \( u \ll w \).

iii. If \( 0 \leq u \ll c \) for each \( c \in \text{int} K \), then \( u = 0 \).

iv. If \( c \in \text{int} K \), \( 0 \leq a_n \) and \( a_n \to 0 \), then there exists \( n_0 \) such that for all \( n > n_0 \), it follows that \( a_n \ll c \).

The following concepts are related with the graph on a metric space.

Let \((X, d)\) be a metric space and \( \Delta \) denote the diagonal of the Cartesian product \( X \times X \). Let \( G \) be a directed graph such that the set \( V(G) \) of its vertices coincides with \( X \) and the set \( E(G) \) of its edges contains all loops; that is, \( E(G) \supseteq \Delta \). Assume that \( G \) has no parallel edges, so one can identify \( G \) with the pair \((V(G), E(G))\). The conversion of a graph \( G \) is denoted by \( G^{-1} \) and which is a graph obtained from \( G \) by reversing the direction of edges. Hence

\[
E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.
\]

By \( \tilde{G} \), we denote the undirected graph obtained from \( G \) by omitting the direction of edges. Indeed, it is more convenient to treat \( \tilde{G} \) as a directed graph for which the set of its edges is symmetric. Under this convention, we have \( E(\tilde{G}) = E(G) \cup E(G^{-1}) \). For any \( x, y \in V', (x, y) \in E' \) such that \( V' \subseteq V(G), E' \subseteq E(G) \), then \((V', E')\) is called a subgraph of \( G \). If \( x \) and \( y \) are vertices in a graph \( G \), then a path from \( x \) to \( y \) of length \( N (N \in \mathbb{N}) \) is a sequence \((x_i)_{i=0}^{N+1}\) of \( N + 1 \) vertices such that \( x_0 = x, x_N = y \) and \((x_{i-1}, x_i) \in E(G)\) for \( i = 1, 2, ..., N \). A graph \( G \) is connected if there is a path between any two vertices. \( G \) is weakly connected if \( \tilde{G} \) is connected. Some basic notations related to connectivity of graphs can be found in [9].

If \( G \) is such that \( E(G) \) is symmetric and \( x \) is a vertex in \( G \), then the subgraph \( G_x \) consisting of all edges and vertices which are contained in some path beginning at \( x \) is called the component of \( G \) containing \( x \). In this case
\[ V(G) = [x]_G \text{ where } [x]_G \text{ denotes the equivalence class of relation } \mathcal{R} \text{ defined on } V(G) \text{ by the rule} \]

\[ y \mathcal{R} z \text{ if there is a path in } G \text{ from } y \text{ to } z. \]

If \( T : X \to X \) is an operator, then we denote by \( F(T) = \{ x \in X : x = Tx \} \) the set of all fixed points of \( T \).

**Definition 2.3** \([6]\) A mapping \( T : X \to X \) is a Banach \( G \)-contraction or simply \( G \)-contraction if the following conditions hold:

i. \( T \) preserves edges of \( G \); \((x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)\) for all \( x, y \in X \),

ii. \( T \) decreases weights of edges of \( G \) if there exists an \( \alpha \in (0, 1) \) such that \((x, y) \in E(G) \Rightarrow d(Tx, Ty) \leq \alpha d(x, y)\) for all \( x, y \in X \).

**Definition 2.4** Let \( K \) be a cone defined as above. A nondecreasing function \( \varphi : \text{int}K \to \text{int}K \), which satisfies the following conditions;

\( \varphi_1. \varphi(\theta) = \theta \) and \( \theta < \varphi(z) < z \) for \( z \in K - \{\theta\} \);

\( \varphi_2. z \in \text{int}K \text{ implies } z - \varphi(z) \in \text{int}K \);

\( \varphi_3. \lim_{n \to \infty} \varphi^n(z) = \theta \) for every \( z \in K - \{\theta\} \);

\( \varphi_4. \sum_{n=0}^{\infty} \varphi^n(z) \text{ converges for all } z \in K - \{\theta\} \);

**Definition 2.5** A mapping \( T : X \to X \) is called orbitally continuous if, for all \( x, y \in X \) and any sequence \( (k_n)_{n \in \mathbb{N}} \) of positive integers, \( T^{k_n}x \to y \) implies \( T(T^{k_n}x) \to Ty \) as \( n \to \infty \).

**Definition 2.6** A mapping \( T : X \to X \) is called orbitally \( G \)-continuous if, for all \( x, y \in X \) and any sequence \( (k_n)_{n \in \mathbb{N}} \) of positive integers, \( T^{k_n}x \to y, (T^{k_n}x, T^{k_{n+1}}x) \in E(G) \) imply \( T(T^{k_n}x) \to Ty \) as \( n \to \infty \).

### 3 Main Results

Throughout the paper we assume that \( X \) is a nonempty set,

\[ X_T = \{ x \in X : (x, Tx) \in E(G) \}, \]

\( G \) is a directed graph and \( E \) is a real Banach space and \( K \) is a cone in \( E \) with \( \text{int}K \neq \emptyset \). By this way, we uniquely determine the limit of a sequence.

Now we define \((G_c, \varphi) -\) contraction in cone metric space with a graph.
Definition 3.1 Let \((X,d)\) be a cone metric space and \(G\) be a graph. The mapping \(T : X \to X\) is called as \((G_c, \varphi)\)−contraction if the following conditions hold:

i. \(T\) preserves the edges of \(G\): \((x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)\) for all \(x, y \in X\),

ii. there exists a function \(\varphi : K \to K\) such that \(d(Tx, Ty) \leq \varphi(d(x, y))\) for all \((x, y) \in E(G)\).

Corollary 3.2 Let \((X,d)\) be a cone metric space endowed with a graph \(G\). If \(T : X \to X\) is a \((G_c, \varphi)\)−contraction, then \(T\) is both a \((G_c^{-1}, \varphi)\)−contraction and a \((\tilde{G}_c, \varphi)\)−contraction.

Theorem 3.3 Let \((X,d)\) be a complete cone metric space and \(G\) be weakly connected. \(T\) is a self mapping on \(X\). We suppose that:

(i) For any sequence \((x_n)_{n \in \mathbb{N}} \in X\) with \(d(x_n, x_{n+1}) \ll c\) for every \(\theta \ll c\) there exist \(k, n_0 \in \mathbb{N}\) such that \((x_{k_n}, x_{k_m}) \in E(G)\) for all \(m, n \in \mathbb{N}\) and \(m, n \geq 0\);

(ii)\(a\) \(T\) is orbitally continuous

or

(ii)\(b\) \(T\) is orbitally \(G\)−continuous and there exists a subsequence \((T^{k_n}x_0)_{k \in \mathbb{N}}\) of \((T^nx_0)_{n \in \mathbb{N}}\) such that \((T^{k_n}x_0, x')_{k \in \mathbb{N}} \in E(G)\) for each \(k \in \mathbb{N}\);

(iii) \(T\) is a \((G_c, \varphi)\)−contraction.

Then \(T\) is a Picard Operator (PO).

Proof 3.4 Take an arbitrary element \(x_0 \in X\) such that \((x_0, Tx_0) \in E(G)\). By using the definition of a \((G_c, \varphi)\)−contraction and an easy induction we get \((T^nx_0, T^{n+1}x_0) \in E(G)\) and \(d(T^nx_0, T^{n+1}x_0) \leq \varphi^n(d(x_0, Tx_0))\) for all \(n \in \mathbb{N}\). Given \(\theta \ll c\) and we choose a positive real number \(\delta\) such that \(c - \varphi(c) + N(\theta + \delta) \subseteq \text{int}K\) where \(N(\theta + \delta) = \{y \in B : \|y\| < \delta\}\). Also choose a natural number \(N\) such that \(\varphi^m(d(x_0, Tx_0)) \ll c - \varphi(c)\) for all \(m \geq N\). Consequently, since \((T^nx_0, T^{m+1}x_0) \in E(G)\) for all \(m \in \mathbb{N}\) then we have \(d(T^nx_0, T^{m+1}x_0) \ll c - \varphi(c)\) for all \(m \geq N\). Fix \(m \geq N\) and we prove

\[
d(T^nx_0, T^{n+1}x_0) \ll c\]

(1)
and from (i) there exists \( m,n+1 \in \mathbb{N} \) such that \( (T^m x_0, T^{n+1} x_0) \in E(G) \) for all \( n \geq m \). If we take \( n = m \), then (1) holds. Now, we assume that (1) holds for some \( n \geq m \). Since \( (T^m x_0, T^{m+1} x_0) \in E(G) \) for any \( m \geq N \), we have that
\[
\begin{align*}
d(T^m x_0, T^{n+2} x_0) & \leq d(T^m x_0, T^{n+1} x_0) + d(T^{m+1} x_0, T^{n+2} x_0) \\
& \leq d(T^m x_0, T^{n+1} x_0) + \varphi(d(T^m x_0, T^{n+1} x_0)) \\
& \ll c - \varphi(c) + \varphi(c) = c.
\end{align*}
\]
Therefore, (1) holds when \( m = n + 1 \). By induction, we deduce (1) holds for all \( m,n \geq N \). Thus, \( \{T^m x_0\}_{m \in \mathbb{N}} \) is a Cauchy sequence in \( X \) and by the completeness of \( X \), \( T^m x_0 \rightarrow x' \in X \) as \( n \rightarrow \infty \). Since \( d(T^n x_0, T^{n+1} x_0) \ll c \), we get \( T^n x_0 \rightarrow x' \) as \( n \rightarrow \infty \). Take an arbitrary element \( x \in X \). Then, if \( (x, x_0) \in E(G) \), then \( (T^n x, T^n x_0) \in E(G) \), for all \( n \in \mathbb{N} \). Hence, from \( \varphi_3 \)
\[
d(T^n x, x') \leq d(T^n x, T^n x_0) + d(T^n x_0, x')
\]
\[
\leq \varphi^n(d(x, x_0)) + d(T^n x_0, x') \ll c.
\]
Letting \( n \rightarrow \infty \), we obtain that \( d(T^n x, x') \ll c \). That is \( T^n x \rightarrow x' \). If \( (x, x_0) \not\in E(G) \), then, since \( G \) is weakly connected, we have a path \( (x_i)_{i=1}^M \) in \( \bar{G} \) from \( x_0 \) to \( x \); that is, \( x_M = x \) and \( (x_{i-1}, x_i) \in E(\bar{G}) \) for \( i = 1, 2, \ldots, M \). With an easy induction we obtain \( (T^n x_{i-1}, T^n x_i) \in E(\bar{G}) \) for \( i = 1, \ldots, M \) and
\[
d(T^n x_0, T^n x) \leq \sum_{i=1}^{M} \varphi^n(d(x_{i-1}, x_i)).
\]
So, letting \( n \rightarrow \infty \) from \( \varphi_3 \) we conclude that \( d(T^n x, x') \ll c \). That is \( T^n x \rightarrow x' \).

Now we are in the position to proved that \( x' \in F(T) \). It is obvious that \( x' \in F(T) \), if (ii) holds. If (ii) occurs since \( d(T^k x_0, x') \ll c \), that is, \( T^k x_0 \rightarrow x' \) and \( (T^k x_0, x') \in E(G) \) for all \( n \in \mathbb{N} \), we attain, using the orbitally \( G \)-continuity of \( T \), that \( d(T^{k+1} x_0, Tx') \ll c \) for all \( n \in \mathbb{N} \). That is, \( T^{k+1} x_0 \rightarrow Tx' \) Thus \( x' = Tx' \). Let \( Ty = y \), for some \( y \in X \), then we have \( T^n y \rightarrow x' \). But it must be the case that \( y = x' \).

The next example illustrates that, in Theorem 3.3, all conditions are necessary for the mapping \( T \) to be a PO.

**Example 3.5** Let \( X = [0, 1] \), \( K = \{x \in E : x \geq 0\} \) and \( E = \mathbb{R}^2 \) with the metric
\[
d : X \times X \rightarrow E
\]
\[
(x, y) \rightarrow d(x, y) = (|x - y|, \alpha |x - y|), \quad \alpha \geq 0.
\]
Consider

\[ E(G) = \{(0, 0)\} \cup \{(0, x) : x \geq 1/2\} \cup \{(x, y) : x, y \in (0, 1]\}, \]

and \( T : X \to X, \)

\[ Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in (0, 1]; \\ 0, & \text{if } x = 0. \end{cases} \]

Then \( G \) is weakly connected, \( X_T \) is nonempty and \( T \) is a \((G_c, \varphi)\)–contraction where \( \varphi(t) = \frac{t}{2} \). Also, \( T \) is both orbitally continuous and orbitally \( G \)–continuous. Thus the conditions of Theorem 3.3 holds; that is, \( T \) is a PO.

There is a close relation between the convergence of iteration sequences, obtained by using the \((G_c, \varphi)\)–contraction and the connectivity of graph \( G \).

**Theorem 3.6** Let \((X, d)\) be a cone metric space endowed with a graph \( G \) and \( T : X \to X \) be a \((G_c, \varphi)\)–contraction, then the following statements are equivalent:

i. \( G \) is weakly connected;

ii. for given \( x, y \in X \), the sequences \((T^n x)_{n \in \mathbb{N}}\) and \((T^n y)_{n \in \mathbb{N}}\) are Cauchy equivalent;

iii. \( \text{card}(F(T)) \leq 1. \)

**Proof 3.7** (i) \( \Rightarrow \) (ii) Let \( T \) be a \((G_c, \varphi)\)–contraction and \( x, y \in X \). By hypothesis, \([x]_{\tilde{G}} = X\), so \( y \in [x]_{\tilde{G}} \). Then there is a path \((x_i)_{i=0}^N \) in \( \tilde{G} \) from \( x \) to \( y \), which means, \( x_0 = x \), \( x_N = y \) and \( (x_{i-1}, x_i) \in E(\tilde{G}) \) for \( i = 1, ..., N \). If we apply an easy induction, we have \((T^n x_{i-1}, T^n x_i) \in E(\tilde{G}) \) for \( i = 1, ..., N \) and

\[ d(T^n x, T^n y) \leq \sum_{i=1}^N \varphi^n (d(x_{i-1}, x_i)) \]

as \( n \to \infty \), from property \( \varphi_3 \), we obtain \( d(T^n x, T^n y) \to 0 \). Likewise, there is a path \((w_i)_{i=0}^M \) in \( \tilde{G} \) from \( x \) to \( Tx \); that is, \( w_0 = x \), \( w_M = Tx \) and \( (w_{i-1}, w_i) \in E(\tilde{G}) \) for \( i = 1, ..., M \). Then by \( \varphi_4 \), the triangle inequality and the definition of \((G_c, \varphi)\)–contraction, we have

\[ d(T^n x, T^{n+1} x) \leq \sum_{i=1}^M \varphi^n (d(w_{i-1}, w_i)). \]

Hence,

\[ \sum_{n=0}^{\infty} d(T^n x, T^{n+1} x) \leq \sum_{i=1}^M \sum_{n=0}^{\infty} \varphi^n (d(w_{i-1}, w_i)) < \infty \]

and this implies that \((T^n x)_{n \in \mathbb{N}}\) is a Cauchy sequence. So, \((T^n y)_{n \in \mathbb{N}}\) is a Cauchy sequence.
(ii) $\Rightarrow$ (iii) Let $T$ be a $(G_c, \varphi)$–contraction and $x, y \in F(T)$. By (ii), $(T^n x)_{n \in \mathbb{N}}$ and $(T^n y)_{n \in \mathbb{N}}$ are Cauchy equivalent, from which one concludes that $x = y$.

(iii) $\Rightarrow$ (i) Conversely, let $G$ is not weakly connected; that is, $\tilde{G}$ is disconnected. Let $x_0 \in X$. Then both the sets $[x_0]_{\tilde{G}}$ and $X - [x_0]_{\tilde{G}}$ are nonempty. Let $y_0 \in X - [x_0]_{\tilde{G}}$ and define

$$Tx = \begin{cases} x_0, & \text{if } x \in [x_0]_{\tilde{G}}, \\ y_0, & \text{if } x \in X - [x_0]_{\tilde{G}}. \end{cases}$$

Obviously, $F(T) = \{x_0, y_0\}$. We prove that $T$ is a $(G_c, \varphi)$–contraction. Let $(x, y) \in E(G)$. Then $[x]_{\tilde{G}} = [y]_{\tilde{G}}$, so either $x, y \in [x_0]_{\tilde{G}}$ or $x, y \in X - [x_0]_{\tilde{G}}$. Hence in both cases $Tx = Ty$, so $(Tx, Ty) \in E(G)$, because $E(G) \supseteq \Delta$, and $d(Tx, Ty) = \theta$. Then, we get $d(Tx, Ty) = 0 \leq \varphi(d(x, y))$. Therefore, $T$ is a $(G_c, \varphi)$–contraction having two fixed points, which conflicts with (iii).

The following result can be obtained from Theorem 3.6 directly.

**Corollary 3.8** Let $(X, d)$ be a complete cone metric space and $G$ be a weakly connected graph. If $T : X \to X$ is a $(G_c, \varphi)$–contraction, then there is $x' \in X$ such that $\lim_{n \to \infty} T^n x = x'$ for all $x \in X$.

**References**


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