On Axiomatic of Inner Product Function

Silvana Liftaj

Department of Mathematics
Faculty of Information Technology
University “A. Moisiu”, Durres, Albania

Eriola Sila

Department of Mathematics
Faculty of Natural Sciences
University of Tirana, Tirana, Albania

Abstract

The study of functions obtained from certain modifications of inner product in a time range was object of study for many mathematicians. Lumer G. has defined the semi-inner product function by replacing the condition of symmetry \((x, y) = (y, x)\) with linearity concerning the variables; Nath B. and Abo Hadi generalized the semi-inner product function by replacing the condition \(11_{\text{yy}} \leq xy \cdot y\) with the condition \(11_{\text{xx}} \leq \left[ x, x \right]^p \cdot \left[ y, y \right]^{p-1}\), where \(1 < p < +\infty\), by introducing semi-inner product function of type \((p)\); Procupanu T. was replace the linearity with parallelogram law; Kramar E. gets Hilbertian semi-norms and H-locally convex spaces avoiding the condition \((x, x) = 0 \) imply \(x = 0\).

In this paper we define two semi-inner product functions by modifying the linearity concerning two variables of inner product and we show the topologies obtained from these modifications are those of a norm function or those of an almost modular function.

Mathematics Subject Classification: 46C50

Keywords: inner product, semi-inner product, semi-norm, almost modular function
1 Introduction

Before the explain the main result of this paper, we introduce some common know concepts.

**Definition 1.1.** The real vectorial space is \((X, +, \cdot)\) which satisfies the following axioms:
1. \( \forall x, y \in X, \; x + y = y + x. \)
2. \( \forall x, y, z \in X, \; (x + y) + z = x + (y + z). \)
3. \( \exists 0 \in X, \; \forall x \in X, x + 0 = x. \)
4. \( \forall x \in X, \exists x' \in X, x + x' = 0. \)
5. \( \forall x \in X, \; 1x = x \) (where 1 \( \in \mathbb{R} \)).
6. \( \forall k, l \in \mathbb{R}, \; \forall x \in X, \; kl(x) = klx. \)
7. \( \forall x, y \in X \) and \( \forall k \in \mathbb{R}, \; k(x + y) = kx + ky. \)
8. \( \forall x \in X \) and \( \forall k, l \in \mathbb{R}, \; (k + l)x = kx + lx. \)

**Definition 1.2.** The normed vectorial space is a \((X, ||||)\), where |||| is map of \(X\) into \(\mathbb{R}\), which satisfies the following axioms:
1. \( \forall x \in X, ||x|| \geq 0 \) and only if \(x = 0. \)
2. \( \forall \lambda \in \mathbb{R}, \forall x \in X, \; ||\lambda x|| = |\lambda| ||x||. \)
3. \( \forall x, y \in X, \; ||x + y|| \leq ||x|| + ||y||. \)

**Definition 1.3.** The pseudo-normed vectorial space is a \((X, ||||)\), where |||| is map of \(X\) into \(\mathbb{R}\), which satisfies the following axioms:
1. \( \forall x \in X, ||x|| \geq 0 \) and only if \(x = 0. \)
2. \( \forall \lambda \in \mathbb{R}, \forall x \in X, \; ||\lambda x|| = |\lambda| ||x||. \)
3. \( \forall x, y \in X, \; ||x + y|| \leq ||x|| + ||y||. \)

**Definition 1.4.** Let \(X\) be a real vectorial space. A inner product in a real vectorial space, which satisfy following conditions \((, ) : X \times X \rightarrow \mathbb{R}\)
1. \((x, x) \geq 0 \) and \((x, x) = 0 \iff x = 0, \forall x \in X. \)
2. \((x, y) = (y, x), \forall (x, y) \in X^2. \)
3. \((\lambda x, y) = \lambda(x, y), \forall (\lambda, x, y) \in \mathbb{R} \times X^2. \)
4. \((x + y, z) = (x, z) + (y, z), \forall (x, y, z) \in X^3. \)

2 Preliminary notes

A basic concept in vectorial spaces is semi-inner product, which is defined by:

**Definition 2.1.** Let \(X\) be a real vector space. A semi-inner product is a mapping \((, ) : X \times X \rightarrow \mathbb{R}\) that satisfies the following properties:
1. \((x, x) \geq 0, \forall x \in X. \)
2. \((x, y) = (y, x), \forall (x, y) \in X^2. \)
3. \((\lambda x, y) = \lambda(x, y), \forall (\lambda, x, y) \in \mathbb{R} \times X^2. \)
4. \((x + y, z) = (x, z) + (y, z), \forall (x, y, z) \in X^3. \)

G. Lumer [6] in 1961 was define the semi-norm function in a space \(X\) as a mapping \([, ] : X \times X \rightarrow \mathbb{R}\) which satisfies the following properties:
1. \([x, x] > 0 \) for \(x \neq 0, \)
2) \([\lambda x, y] = \lambda [x, y]\),
3) \([x + y, z] = [x, z] + [y, z]\),
4) \([x, y]^2 \leq [x, x] \cdot [y, y]\).

In this definition conditions 2) and 3) required to meet only for the first variable of an inner production and forego from the condition of symmetry \((x, y) = (y, x)\). Regardless of those modifications in topological aspect we obtain the normed spaces by setting \([x] = \sqrt{[x, x]}\). A case study of spaces obtained by a semi-inner product function is carry out by J. R. Giles [2] in 1967 and T. Husain together with B. Malviya [3] in 1972. B. Nath [7] in 1971 generalizes the semi-inner product function of G. Lumer [6] by defining the generalized semi-inner product function as a mapping \([, ] : X \times X \to \mathbb{R}\) which satisfies the following properties:
1) \([x, x] > 0\) for \(x \neq 0\),
2) \([\lambda x, y] = \lambda [x, y]\),
3) \([x + y, z] = [x, z] + [y, z]\),
4) \([x, y] \leq [x, x]^{\frac{1}{p}} \cdot [y, y]^{\frac{p-1}{p}}\), where \(p \in ]0, +\infty[\).

This function which is called semi-inner product of type \((p)\) defines a norm function by setting \([x] = \sqrt{[x, x]}\). In 2010 H. Zhang [9] generalize further the semi-inner product function with condition \([x, y] \leq \varphi [x, x] \psi [y, y]\), where \(\varphi\) and \(\psi\) are two specific function of \(\mathbb{R}^+\) to \(\mathbb{R}^+\). If \(\varphi (t) = \psi (t) = \sqrt{t}\) we obtain the semi-inner product function of G. Lumer [6]. Further, if \(\varphi (t) = t^p\) and \(\psi (t) = t^{\frac{p-1}{p}}\) we obtain the semi-inner product function of type \((p)\).

For \(\varphi\) and \(\psi\) that satisfies the condition \(\varphi (t) \psi (t) = t\) we take the concrete shapes of semi-inner product functions. E. Kramer [5] in 1981 studied H-locally convex spaces which obtained from Hilbertian semi-norms generated by a collection of semi-inner products satisfying the following inner production properties:

a) \((x, x) > 0\) for all \(x \neq 0\),
b) \((x, y) = (y, x)\) for all \((x, y) \in X^2\),
c) \((\lambda x, y) = \lambda (x, y)\) for all \((\lambda, x, y) \in \mathbb{R} \times X^2\),
d) \((x + y, z) = (x, z) + (y, z)\) for all \((x, y, z) \in X^3\).

### 3 Main results

In this paper we will treat two cases of generalization of the inner product functions.

**Definition 3.1.** The function \((, ) : X \times X \to \mathbb{R}\) is called (a)-pseudo inner production if it satisfy the properties:
1) \((x, x) \geq 0\) for all \(x \in X\) and \((x, x) = 0 \iff x = 0\),
2) \((x, y) = (y, x), \forall (x, y) \in X^2\),
3) \([\lambda x, y] = \lambda [x, y], \forall \lambda, x, y \in \mathbb{R} \times X \times X\),
4) \(|x_1 + x_2, y| \leq \left(\sqrt{(x_1, x_1)} + \sqrt{(x_2, x_2)}\right) \sqrt{(y, y)}, \ \forall \ (x_1, x_2, y) \in X^3.\)

Let illustrate this by an example.

**Example 3.1.** Let \((X, \rho)\) be a normed space. From Hanh-Banach Theorem for all \(x \neq 0\) there is a continuous linear form \(f_x: X \rightarrow \mathbb{R}\) such that \(f_x(x) = x.\)

For \(x = 0, \ f_{x=0} = 0\), we have \(|f_x| \leq 1.\) Note \((x, y) = f_x(y) f_x(x).\) We construct the function \((\ , \ ) : X \times X \rightarrow \mathbb{R}\) defined by

\[(x, y) = f_x(y) f_x(x), \text{ for all } (x, y) \in X^2.\]

We have:

- \((x, x) = f_x(x) f_x(x) = f_x^2(x) = \rho^2(x) \geq 0.\)
- \((x, y) = f_x(y) f_x(x) = f_x^2(y) = (y, x), \text{ for all } (x, y) \in X^2.\)

\[|\lambda x, y| = |f_{\lambda x}(y) f_\lambda(x)| = |f_x(y)| \rho(\lambda x) \leq ||f_x|| \rho(y) ||f_\lambda\| \rho(x) \leq |\lambda| \rho(x) \rho(y) \leq |\lambda| \sqrt{(x, x)} \sqrt{(y, y)}, \text{ because } (x, x) = \rho^2(x), \text{ hereof } \rho(x) = \sqrt{(x, x)}.\]

- \(|(x_1 + x_2, y)| = |f_{x_1+x_2}(y) f_{x_1}(x_1) + f_{x_2}(x_2)| = |f_{x_1}(x_1) + f_{x_2}(x_2)| \leq ||f_{x_1}|| \rho(x_1 + x_2) \leq \rho(y) \rho(x_1 + x_2) \leq \rho(p(x_1) + p(x_2)) \rho(y) = \sqrt{(x_1, x_1)} + \sqrt{(x_2, x_2)} \sqrt{(y, y)}.\]

The satisfying of above properties shows that the above function is (a)-pseudo-inner product function.

**Proposition 3.1.** If \((\ , \ ) : X \times X \rightarrow \mathbb{R}\) is (a)-pseudo inner product, then we can define a norm.

**Proof.** Let denote \(q(x) = \sqrt{(x, x)}\) for all \(x \in X.\) The function \(q : X \rightarrow \mathbb{R}^+\) satisfies the following properties:

- \(q(x) \geq 0\) for all \(x \in X\) and \(q(x) = 0 \iff x = 0\) are clear.
- \(q(\lambda x) = |\lambda| q(x), \text{ for all } (\lambda, x) \in \mathbb{R} \times X.\)

Indeed, for all \((\lambda, x) \in \mathbb{R} \times X\) we have \((\lambda x, \lambda x) \leq |\lambda| \sqrt{(x, x)} \sqrt{(\lambda x, \lambda x)}\) from which we get \(\sqrt{(\lambda x, \lambda x)} \leq |\lambda| \sqrt{(x, x)}, \text{ thus}\)

\[q(\lambda x) \leq |\lambda| q(x) \text{ (1)}\]

On the other hand, for \(\lambda = 0\), we have \(q(0 x) = q(0) = 0 = 0 q(x).\)

For \(\lambda \neq 0, q(x) = q\left(\lambda \cdot \frac{1}{\lambda} x\right) \leq \frac{1}{|\lambda|} q(\lambda x), \text{ then} \)

\[|\lambda| q(x) \leq q(\lambda x) \text{ (2)}\]

From (1) and (2) we imply \(q(\lambda x) = |\lambda| q(x), \text{ for all } (\lambda, x) \in \mathbb{R} \times X.\)

- \(q(x_1 + x_2) \leq q(x_1) + q(x_2)\), for all \(x_1, x_2 \in X,\)

\[
(x_1 + x_2, x_1 + x_2) \leq \left[\sqrt{(x_1, x_1)} + \sqrt{(x_2, x_2)}\right] \sqrt{(x_1 + x_2, x_1 + x_2)} \text{ from where we get:}
\]

\[
\sqrt{(x_1 + x_2, x_1 + x_2)} \leq \sqrt{(x_1, x_1)} + \sqrt{(x_2, x_2)}, \text{ thus}
\]

\[
q(x_1 + x_2) \leq q(x_1) + q(x_2),
\]

So, the function \(q\) is a norm function.

**Definition 3.2.** The function \((\ , \ ) : X \times X \rightarrow \mathbb{R}\) is called (b)-pseudo-inner product if it satisfy the following properties:
1) \((x, x) \geq 0\) for all \(x \in X\) and \((x, x) = 0 \iff x = 0\)
3) \(|(\lambda x, y)| \leq (x, y)\), for all \(x, y \in X\) and \(|\lambda| \leq 1\)
2) \((x, y) = (y, x), \forall (x, y) \in X^2\)
4) \(\sqrt{(x+y, x+y)} \leq \sqrt{(x,x)} + \sqrt{(y,y)}\), \(\forall (x, y) \in X^2\)

Let illustrate this by an example.

**Example 3.2.** Let \(X = \mathbb{R}\) and \((x, y) = \frac{x \cdot y}{1 + |x \cdot y|}\). We have

- \((x, x) = \frac{x^2}{1 + x^2} \geq 0\) and \((x, x) = 0 \iff x = 0\).
- \(|(\lambda x, y)| = \frac{|\lambda| \cdot |xy|}{1 + |\lambda| \cdot |xy|} \leq \frac{|xy|}{1 + |xy|}\), for all \((x, y) \in \mathbb{R}^2\), because the function \(f: t \to \frac{t}{1+t}, t \in \mathbb{R}\) is increased for \(t > 0\).
- \((x, y) = (y, x)\) is clear for all \((x, y) \in \mathbb{R}^2\).
- \((x+y, x+y) = \frac{(x+y)^2}{1+(x+y)^2} = \frac{x^2 + 2|x||y| + y^2}{1 + x^2 + 2|x||y| + y^2} \leq \frac{x^2}{1 + x^2} + \frac{2|x||y|}{1 + x^2 + y^2}\)

From where we get

\[\sqrt{(x+y, x+y)} \leq \sqrt{(x,x)} + \sqrt{(y,y)}.\]

The satisfying of above properties shows that the above function is (b)-pseudo-inner product function.

**Proposition 3.2.** If \((, ) : X \times X \to \mathbb{R}\) is (b)-pseudo-inner product, then the function \(q : X \to \mathbb{R}^+\) such that \(q(x) = \sqrt{(x,x)}\) is a quasi modular function.

**Proof.**
- \(q(x) \geq 0\) for all \(x \in X\) and \(q(x) = 0 \iff x = 0\) are clear.
- \(q(\lambda x) = |\lambda| q(x), \forall |\lambda| \leq 1, x \in X\).
- \(q(x_1, x_2) = \sqrt{(x_1 + x_2, x_1 + x_2)} \leq \sqrt{(x_1, x_1) + (x_2, x_2)} = q(x_1) + q(x_2), \forall (x_1, x_2) \in X^2\)

From \(q(\lambda x) \leq q(x)\) for \(|\lambda| \leq 1, x \in X\) we imply \(q(-x) \leq q(x), q(x) = q(-(-x)) \leq q(-x)\)
we get \( q(-x) = q(x) \), for all \( x \in X \).

- On the other hand \( \forall x \in X \) and \( \alpha > 0, \beta > 0 \) such that \( \alpha + \beta = 1 \) we get the satisfaction of condition \( q(\alpha x + \beta y) \leq q(\alpha x) + q(\beta y) \leq q(x) + q(y) \).

Thus \( q \) is a quasi modular function.

**Proposition 3.3.** For any vector space in which is defined a \((b)\)-pseudo inner product function, the set \( X_0 = \{ x \in X \mid \lim_{\lambda \rightarrow 0} q(\lambda x) = 0 \} \) is a vectorial subspace which coincides with the space itself.

**Proof.** It is clear that \( X_0 \subset X \). Let be \( x_1 \) and \( x_2 \) from \( X_0 \). Since \( x_1 \in X_0 \) and \( x_2 \in X_0 \) we have \( \lim_{\lambda \rightarrow 0} q(\lambda x_1) = 0 \), \( \lim_{\lambda \rightarrow 0} q(\lambda x_2) = 0 \), from which we get:

\[
0 \leq q(\lambda(x_1 + x_2)) \leq q(\lambda x_1) + q(\lambda x_2) \quad \lim_{\lambda \rightarrow 0} \rightarrow 0.
\]

So,

\[
\lim_{\lambda \rightarrow 0} q(\lambda(x_1 + x_2)) = 0 \Rightarrow x_1 + x_2 \in X_0.
\]

For each \( \mu \in \mathbb{R}^+ \), \( x \in X_0 \Rightarrow \mu x \in X_0 \)

\[
0 \leq q(\lambda(\mu x)) \leq |\lambda| q(\mu x) \quad \lim_{\lambda \rightarrow 0} \rightarrow 0,
\]

since \( \lambda \rightarrow 0 \Rightarrow |\lambda| < 1 \) and \( |\lambda| \rightarrow 0 \), so \( \mu x \in X_0 \).

Thus, \( X_0 \) is a subspace of the vectorial space \( X \). From the other hand, for all \( x \in X \) we have \( |\lambda|q(x) \rightarrow 0 \), so \( q(\lambda x) \leq |\lambda|q(x) \rightarrow 0 \), from which \( x \in X_0 \). Thus, \( X = X_0 \).

**Proposition 3.4.** The function \( p : X \rightarrow \mathbb{R}^+ \), where \( p(x) = \inf\{ \lambda > 0 \mid q\left(\frac{x}{\lambda}\right) < 1\} \) is the semi-norm.

**Proof.**

- It is clear that \( p(x) \geq 0 \), for all \( x \in X \) and \( p(0) = 0 \).

- For \( k = 0 \), we have \( p(0) = 0 \); for \( k \neq 0 \) we have \( p(kx) = \inf\{ \lambda > 0 \mid q\left(\frac{kx}{\lambda}\right) < 1\} = \inf\{ \lambda > 0 \mid q\left(\frac{kx}{\lambda}\right) < 1\} = \inf\{ \lambda > 0 \mid q\left(\frac{x}{k\lambda}\right) < 1\} = k \cdot p(x) \)

- \( p(x + y) = \inf\{ \lambda > 0 \mid q\left(\frac{x+y}{\lambda}\right) < 1\} \).

Let mark \( A = \{ \lambda > 0 \mid q\left(\frac{x}{\lambda}\right) < 1\} \), \( B = \{ \lambda > 0 \mid q\left(\frac{y}{\lambda}\right) < 1\} \),

\( C = \{ \lambda > 0 \mid q\left(\frac{x+y}{\lambda}\right) < 1\} \).

We will show that \( A + B \subset C \).
Indeed, let be \( \lambda_1 \in A, \lambda_2 \in B \), where \( q \left( \frac{x}{\lambda_1} \right) < 1, q \left( \frac{y}{\lambda_2} \right) < 1 \). Let mark \( \lambda_0 = \lambda_1 + \lambda_2 \). We have

\[
q \left( \frac{x+y}{\lambda_0} \right) = q \left( \frac{\lambda_1 x + \lambda_2 y}{\lambda_1 + \lambda_2} \right) \leq \frac{\lambda_1}{\lambda_1 + \lambda_2} q \left( \frac{x}{\lambda_1} \right) + \frac{\lambda_2}{\lambda_1 + \lambda_2} q \left( \frac{y}{\lambda_2} \right) < \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2} = 1,
\]

so \( \lambda_1 + \lambda_2 \in C \).

We have that

\[
A + B \subset C \Rightarrow \inf C \leq \inf (A + B) = \inf A + \inf B = p(x+y) \leq p(x) + p(y).
\]

**Proposition 3.5.** The function \( d : X^2 \to \mathbb{R}^+ \), such that \( d(x, y) = q(x - y) \) is a distance function.

**Proof.**

- \( d(x, y) = q(x - y) \geq 0 \) for all \( (x, y) \in X^2 \) and
  \( d(x, y) = q(x - y) = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y \).
- \( d(x, y) = q(x - y) = q(y - x) = d(y, x) \), because \( q(-x) = q(x) \).
- \( d(x, y) = q(x - y) = q(x - z + z - y) \leq q(x - z) + q(z - y) = d(x, z) + d(z, y) \),
  \( \forall (x, y, z) \in X^3 \),

Thus \( d \) is the distance function.

**Remark 3.1.** It is true the implication \( p(x) < 1 \Rightarrow q(x) < 1 \).

**Proof.** From the property of inferior there is \( \lambda_1 > 0 \) such that \( p(x) < \lambda_1 < 1 \) and

\[
q \left( \frac{x}{\lambda_1} \right) < 1, \text{ from which } q(x) = q \left( \lambda_1 \frac{x}{\lambda_1} \right) \leq |\lambda_1| \left| q \left( \frac{x}{\lambda_1} \right) \right| < 1 \cdot 1 = 1. \text{ Thus,}
\]

\[
p(x) < 1 \Rightarrow q(x) < 1.
\]

**Proposition 3.6.** In any vector space equipped with a (b)-pseudo inner product the topology of semi-norm \( p \) is equivalent to the topology \( T_d \) of the distance function.

**Proof.** Let be \( x_0 \) whatever from \( X \). We mark \( S(x_0, \varepsilon) = \{ x \in X \mid q(x - x_0) < \varepsilon \} \) and \( M(x_0, \varepsilon) = \{ x \in X \mid p(x - x_0) < \varepsilon \} \). We note that \( M(x_0, \frac{\varepsilon}{2}) \subset S(x_0, \frac{\varepsilon}{2}) \subset M(x_0, \varepsilon) \).

Indeed, if \( x \in M(x_0, \frac{\varepsilon}{2}) \Rightarrow p(x - x_0) < \frac{\varepsilon}{2} \Rightarrow p \left( \frac{x-x_0}{\frac{\varepsilon}{2}} \right) < 1 \Rightarrow q \left( \frac{x-x_0}{\frac{\varepsilon}{2}} \right) < 1 \)

\[
\Rightarrow q(x-x_0) = q \left( \frac{\varepsilon}{2} \frac{x-x_0}{\frac{\varepsilon}{2}} \right) \leq \left| \frac{\varepsilon}{2} \right| q \left( \frac{x-x_0}{\frac{\varepsilon}{2}} \right) < \left| \frac{\varepsilon}{2} \right| \cdot 1 = \frac{\varepsilon}{2} \Rightarrow x \in S(x_0, \frac{\varepsilon}{2}).
\]

On the other hand, if \( x \in S(x_0, \frac{\varepsilon}{2}) \Rightarrow q(x - x_0) < \frac{\varepsilon}{2} \Rightarrow q \left( \frac{x-x_0}{\frac{\varepsilon}{2}} \right) < \frac{1}{\frac{\varepsilon}{2}} \Rightarrow q(x - x_0) < 1 \).
\[
\frac{2 \varepsilon}{2} < 1 \Rightarrow p(x - x_0) = \inf\{\lambda > 0 \mid q\left(\frac{x - x_0}{\lambda}\right) < 1\} \leq \frac{\varepsilon}{2} < \varepsilon \Rightarrow x \in M(x_0, \varepsilon).
\]
So,
\[
M(x_0, \frac{\varepsilon}{2}) \subset S(x_0, \frac{\varepsilon}{2}) \subset M(x_0, \varepsilon).
\]

References


Received: November 1, 2013