A Note on the lambda-Daehee Polynomials

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Abstract. Recently, Daehee polynomials are introduced in [6]. In this paper, we study the \( \lambda \)-Daehee polynomials and investigate their properties arising from the \( p \)-adic integral equations.

1. Introduction

The Daehee polynomials of the first kind are defined by the generating function to be

\[
\left( \frac{\log (1 + t)}{t} \right) (1 + t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!},
\]

(see [6]) and the Daehee polynomials of the second kind are given by

\[
\left( \frac{\log (1 + t)}{t} \right) (1 + t)^{x+1} = \sum_{n=0}^{\infty} \hat{D}_n(x) \frac{t^n}{n!},
\]

(see [6]).

As is well known, the Bernoulli polynomials of order \( k (\in \mathbb{N}) \) are defined by the generating function to be

\[
\left( \frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!},
\]

(see [7-15]).

When \( x = 0 \), \( B_n^{(k)} = B_n^{(k)}(0) \) are the Bernoulli numbers of order \( k \). In particular, if \( k = 1 \), \( B_n(x) = B_n^{(1)}(x) \) are the Bernoulli polynomials.

When \( x = 0 \), \( B_n = B_n(0) \) are the Bernoulli numbers.

Throughout this paper, \( \mathbb{Z}_p, \mathbb{Q}_p \) and \( \mathbb{C}_p \) will denote the ring of \( p \)-adic integers, the field of \( p \)-adic numbers and the completion of algebraic closure of \( \mathbb{Q}_p \).

Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable functions on \( \mathbb{Z}_p \). For \( f \in UD(\mathbb{Z}_p) \), the bosonic \( p \)-adic integral on \( \mathbb{Z}_p \) is defined by

\[
I(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x),
\]

(see [9]).

Let \( f_1 \) be the translation of \( f \) with \( f_1(x) = f(x + 1) \). Then, by (4), we get

\[
I(f_1) = I(f) + f'(0),
\]

where \( f'(0) = \frac{df(x)}{dx} \bigg|_{x=0} \) (see [1-9]).
A note on the lambda-Daehee polynomials

The Stirling number of the first kind is given by

\[(x)_n = x(x - 1) \cdots (x - n + 1) = \sum_{l=0}^{n} S_1(n, l) x^l, \quad (n \geq 0),\]

and the Stirling number of the second kind is defined by the generating function to be

\[(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!},\]

(see [13]).

In this paper, we consider the \(\lambda\)-Daehee polynomials which are derived from the bosonic \(p\)-adic integral on \(\mathbb{Z}_p\). Finally, we investigate some properties on the \(\lambda\)-Daehee polynomials related to Bernoulli polynomials of order \(k\) \((k \in \mathbb{N})\).

2. On the \(\lambda\)-Daehee Polynomials

In this section, we assume that \(t \in \mathbb{C}_p\) with \(|t|_p < p^{-\frac{1}{p-1}}\) and \(\lambda \in \mathbb{Z}_p\).

Now, we consider \(\lambda\)-Daehee polynomials which are a generalization of Daehee polynomials, defined as follows:

\[
\frac{\lambda \log (1 + t)}{(1 + t)^\lambda - 1} (1 + t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!}.
\]

(8)

When \(x = 0\), \(D_{n,\lambda} = D_{n,\lambda}(0)\) are called the \(\lambda\)-Daehee numbers. It is easy to see that \(D_n(x) = D_{n,1}(x)\) and \(D_n = D_{n,1}\).

Let us take \(f(x) = (1 + t)^{\lambda x}\). From (5), we have

\[
\int_{\mathbb{Z}_p} (1 + t)^{\lambda x} d\mu_0(y) = \frac{\lambda \log (1 + t)}{(1 + t)^\lambda - 1}.
\]

(9)

Thus, by (9), we get

\[
\int_{\mathbb{Z}_p} (1 + t)^{\lambda y + x} d\mu_0(y) = \frac{\lambda \log (1 + t)}{(1 + t)^\lambda - 1} (1 + t)^x
\]

\[
= \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!},
\]

(10)

and

\[
\int_{\mathbb{Z}_p} (1 + t)^{\lambda y + x} d\mu_0(y) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left( \frac{\lambda y + x}{n} \right) d\mu_0(y) t^n
\]

\[
= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (\lambda y + x)_n d\mu_0(y) \frac{t^n}{n!}.
\]

(11)

Therefore, by (10) and (11), we obtain the following theorem.
Theorem 1. For \( n \geq 0 \), we have

\[
D_{n, \lambda} (x) = \int_{\mathbb{Z}_p} (\lambda y + x)_n \, d\mu_0 (y).
\]

By (8), we get

\[
\frac{\lambda t}{e^{\lambda t} - 1} e^{(\lambda t)x} = \sum_{n=0}^{\infty} D_{n, \lambda} (x) \frac{1}{n!} (e^t - 1)^n \tag{12}
\]

and

\[
\frac{\lambda t}{e^{\lambda t} - 1} e^{(\lambda t)x} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} S_2 (m, n) D_{n, \lambda} (x) \right) \frac{t^m}{m!} \tag{13}
\]

From (6) and Theorem 1, we note that

\[
D_{n, \lambda} (x) = \int_{\mathbb{Z}_p} (\lambda y + x)_n \, d\mu_0 (y) \tag{14}
\]

= \sum_{l=0}^{n} S_1 (n, l) \, \lambda^l \int_{\mathbb{Z}_p} \left( y + \frac{x}{\lambda} \right)^l \, d\mu_0 (y)

= \sum_{l=0}^{n} S_1 (n, l) \, \lambda^l B_l \left( \frac{x}{\lambda} \right).

Therefore, by (12), (13) and (14), we obtain the following theorem.

Theorem 2. For \( m \geq 0 \), we have

\[
D_{m, \lambda} (x) = \sum_{l=0}^{m} S_1 (m, l) \, \lambda^l B_l \left( \frac{x}{\lambda} \right),
\]

and

\[
\lambda^m B_m \left( \frac{x}{\lambda} \right) = \sum_{n=0}^{m} S_2 (m, n) \, D_{n, \lambda} (x).
\]

Let us consider the \( \lambda \)-Daehee polynomials of the first kind with order \( k (\in \mathbb{N}) \) as follows:

\[
D_{n, \lambda}^{(k)} (x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1 + \cdots + \lambda x_k + x)_n \, d\mu_0 (x_1) \cdots d\mu_0 (x_k). \tag{15}
\]

From (15), we can derive the generating function of \( D_{n, \lambda}^{(k)} (x) \) as follows:
\[
\sum_{n=0}^{\infty} D^{(k)}_{n,\lambda}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1 + \cdots + \lambda x_k + x) d\mu_0(x_1) \cdots d\mu_0(x_k) \frac{t^n}{n!} \\
= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda x_1 + \cdots + \lambda x_k + x} d\mu_0(x_1) \cdots d\mu_0(x_k)
= \left( \frac{\lambda \log (1+t) - 1}{(1+t)^\lambda - 1} \right)^k (1+t)^x.
\]

By (15), we easily get
\[
D^{(k)}_{n,\lambda}(x)
= \sum_{l=0}^{n} S_1(n, l) \lambda^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( x_1 + \cdots + x_k + \frac{x}{\lambda} \right)^l d\mu_0(x_1) \cdots d\mu_0(x_k)
= \sum_{l=0}^{n} S_1(n, l) \lambda^l B_l^{(k)}\left( \frac{x}{\lambda} \right).
\]

From (16), we note that
\[
\left( \frac{\lambda t}{e^{\lambda t} - 1} \right)^k e(t) \lambda^t = \sum_{n=0}^{\infty} D^{(k)}_{n,\lambda}(x) \frac{1}{n!} \left( e^t - 1 \right)^n
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} D^{(k)}_{n,\lambda}(x) S_2(m, n) \right) \frac{1}{m!} \frac{t^m}{m!},
\]
and
\[
\left( \frac{\lambda t}{e^{\lambda t} - 1} \right)^k e(t) \lambda^t = \sum_{m=0}^{\infty} \lambda^m B_m^{(k)}\left( \frac{x}{\lambda} \right) \frac{1}{m!} \frac{t^m}{m!}.
\]

Therefore, by (17), (18) and (19), we obtain the following theorem.

**Theorem 3.** For \( m \geq 0, \ k \in \mathbb{N} \), we have
\[
D^{(k)}_{n,\lambda}(x) = \sum_{l=0}^{n} S_1(n, l) \lambda^l B_l^{(k)}\left( \frac{x}{\lambda} \right),
\]
and
\[
\lambda^m B_m^{(k)}\left( \frac{x}{\lambda} \right) = \sum_{n=0}^{m} S_2(m, n) D^{(k)}_{n,\lambda}(x).
\]
For \( n \geq 0 \), the rising factorial sequence is defined by
\[
x^{(n)} = x(x + 1)\cdots(x + n - 1) = (-1)^n (-x)_n
\] (20)
\[
= \sum_{l=0}^{n} (-1)^{n-l} S_1(n, l) x^l.
\]
Let us define the \( \lambda \)-Dahee polynomials of the second kind as follows:
\[
\frac{\lambda \log (1 + t)}{(1 + t)^\lambda - 1} (1 + t)^{\lambda + x} = \sum_{n=0}^{\infty} \hat{D}_{n, \lambda}(x) \frac{t^n}{n!}.
\] (21)

Note that \( \hat{D}_{n, 1}(x) = \hat{D}_n(x) \).

Let us take \( f(x) = (1 + t)^{-\lambda x} \). Then, by (5), we get
\[
\int_{\mathbb{Z}_p} (1 + t)^{-\lambda x} d\mu_0(x) = \frac{\lambda \log (1 + t)}{(1 + t)^\lambda - 1} (1 + t)^\lambda.
\] (22)

Thus, from (22), we have
\[
\int_{\mathbb{Z}_p} (1 + t)^{-\lambda y + x} d\mu_0(y) = \sum_{n=0}^{\infty} \hat{D}_{n, \lambda}(x) \frac{t^n}{n!},
\] (23)

and
\[
\int_{\mathbb{Z}_p} (1 + t)^{-\lambda y + x} d\mu_0(y) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left( -\frac{\lambda y + x}{n} \right) d\mu_0(y) t^n
\] (24)
\[
= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (-\lambda y + x) d\mu_0(y) \frac{t^n}{n!}.
\]

By (23) and (24), we get
\[
\hat{D}_{n, \lambda}(x) = \int_{\mathbb{Z}_p} (-\lambda y + x) d\mu_0(y)
\] (25)
\[
= \sum_{l=0}^{n} S_1(n, l) (-\lambda)^l \int_{\mathbb{Z}_p} \left( y - \frac{x}{\lambda} \right)^l d\mu_0(y)
\]
\[
= \sum_{l=0}^{n} S_1(n, l) (-\lambda)^l B_l \left( \frac{x}{\lambda} \right).
\]

It is well known that \( B_n(1 - x) = (-1)^n B_n(x) \).

From (25), we have
\[
\hat{D}_{n, \lambda}(x) = \sum_{l=0}^{n} S_1(n, l) \lambda^l B_l \left( 1 + \frac{x}{\lambda} \right).
\] (26)
By (21), we get
\[
\frac{\lambda t}{e^{\lambda t} - 1} e^{(\lambda + x)t} = \sum_{n=0}^{\infty} \hat{D}_{n,\lambda}(x) \frac{(e^t - 1)^n}{n!}
\]
(27)
\[= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \hat{D}_{n,\lambda}(x) S_2(m, n) \right) \frac{t^m}{m!},
\]
and
\[
\frac{\lambda t}{e^{\lambda t} - 1} e^{(\lambda + x)t} = \frac{\lambda t}{e^{\lambda t} - 1} e^{(1 + \frac{x}{\lambda})t}
\]
(28)
\[= \sum_{m=0}^{\infty} \lambda^m B_m \left( 1 + \frac{x}{\lambda} \right) \frac{t^m}{m!}.
\]

Therefore, by (26), (27) and (28), we obtain the following theorem.

**Theorem 4.** For $m \geq 0$, we have
\[
\hat{D}_{m,\lambda}(x) = \sum_{l=0}^{m} S_1(m, l) \lambda^l B_l \left( 1 + \frac{x}{\lambda} \right)
\]
and
\[
\lambda^m B_m \left( 1 + \frac{x}{\lambda} \right) = \sum_{n=0}^{m} S_2(m, n) \hat{D}_{n,\lambda}(x).
\]

**Remark.** When $x = 0$, we get
\[
\lambda^m B_m (1) = \sum_{n=0}^{m} \hat{D}_{n,\lambda} S_2(m, n).
\]

For $\lambda = 1$, we have
\[
B_m (1) = \sum_{n=0}^{m} \hat{D}_{n,\lambda} S_2(m, n).
\]

For $k \in \mathbb{N}$, we define the $\lambda$-Daehee polynomials of the second kind with order $k$ :
\[
\hat{D}_{n,\lambda}^{(k)}(x) = \int_{Z_p} \cdots \int_{Z_p} (- (\lambda x_1 + \cdots + \lambda x_k) + x)_n \, d\mu_0(x_1) \cdots d\mu_0(x_k).
\]
(29)
From (29), we can derive the generating function of $\hat{D}_{n,\lambda}^{(k)}(x)$ as follows:

\[
\sum_{n=0}^{\infty} \hat{D}_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}
= \sum_{n=0}^{\infty} \int_{Z_p} \cdots \int_{Z_p} \left( -\lambda x_1 + \cdots + \lambda x_k + x \right)_n d\mu_0(x_1) \cdots d\mu_0(x_k) \frac{t^n}{n!}
= \int_{Z_p} \cdots \int_{Z_p} (1 + t)^{-\left(\lambda x_1 + \cdots + \lambda x_k + x \right)} d\mu_0(x_1) \cdots d\mu_0(x_k)
= \left( \frac{\lambda \log(1 + t)}{(1 + t)^\lambda - 1} \right)^k (1 + t)^{\lambda k + x}.
\]

Thus, by (30), we get

\[
\sum_{n=0}^{\infty} \hat{D}_{n,\lambda}^{(k)}(x) \frac{(e^t - 1)^n}{n!} = \left( \frac{\lambda t}{e^{\lambda t} - 1} \right)^k e^{(\lambda k + x)t}
= \sum_{m=0}^{\infty} \lambda^m B_m^{(k)} \left( k + \frac{x}{\lambda} \right) \frac{t^m}{m!},
\]

and

\[
\sum_{n=0}^{\infty} \hat{D}_{n,\lambda}^{(k)}(x) \frac{(e^t - 1)^n}{n!}
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} S_2(m, n) \hat{D}_{n,\lambda}^{(k)}(x) \right) \frac{t^m}{m!}.
\]

By (29), we easily get

\[
\hat{D}_{m,\lambda}^{(k)}(x)
= \sum_{l=0}^{m} S_1(m, l) \lambda^l \left( -1 \right)^l \int_{Z_p} \cdots \int_{Z_p} \left( x_1 + \cdots + x_k - \frac{x}{\lambda} \right)^l d\mu_0(x_1) \cdots d\mu_0(x_k)
= \sum_{l=0}^{m} S_1(m, l) \lambda^l \left( -1 \right)^l B_l^{(k)} \left( -\frac{x}{\lambda} \right).
\]
A note on the lambda-Daehee polynomials

We observe that

\[
\sum_{n=0}^{\infty} B_n^{(k)} (k - x) \frac{t^n}{n!} = \left( \frac{t}{e^t - 1} \right)^k e^{(k-x)t} = \left( \frac{-t}{e^{-t} - 1} \right)^k e^{-xt} = \sum_{n=0}^{\infty} (-1)^n B_n^{(k)} (x) \frac{t^n}{n!}.
\]

(34)

Thus, by (34), we get

\[
B_n^{(k)} (k - x) = (-1)^n B_n^{(k)} (x).
\]

(35)

From (33) and (35), we have

\[
\hat{D}^{(k)}_{m,\lambda} (x) = \sum_{l=0}^{m} S_1 (m, l) (-\lambda)^l B_l^{(k)} \left( -\frac{x}{\lambda} \right) = \sum_{l=0}^{m} S_1 (m, l) \lambda^l B_l^{(k)} \left( k + \frac{x}{\lambda} \right).
\]

(36)

Therefore, by (31), (32) and (36), we obtain the following theorem.

**Theorem 5.** For \( m \geq 0 \), we have

\[
\hat{D}^{(k)}_{m,\lambda} (x) = \sum_{l=0}^{m} S_1 (m, l) \lambda^l B_l^{(k)} \left( k + \frac{x}{\lambda} \right),
\]

and

\[
\lambda^m B_m^{(k)} \left( k + \frac{x}{\lambda} \right) = \sum_{n=0}^{m} S_2 (m, n) \hat{D}^{(k)}_{n,\lambda} (x).
\]

We observe that
\[
(-1)^n \frac{D_{n,\lambda}(x)}{n!} = \int_{\mathbb{Z}_p} \binom{x + \lambda y}{n} d\mu_0(y)
\] 

\[
= (-1)^n \int_{\mathbb{Z}_p} \binom{-y\lambda - x + n - 1}{n} d\mu_0(y)
\]

\[
= \sum_{m=0}^{n} \binom{n-1}{n-m} \frac{1}{m!} \int_{\mathbb{Z}_p} \binom{-y\lambda - x}{m} d\mu_0(y)
\]

\[
= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{1}{m!} \int_{\mathbb{Z}_p} \binom{-y\lambda - x}{m} d\mu_0(y)
\]

\[
= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{D_{m,\lambda}(-x)}{m!},
\]

and

\[
(-1)^n \frac{\hat{D}_{n,\lambda}(x)}{n!} = (-1)^n \int_{\mathbb{Z}_p} \binom{-\lambda y + x}{n} d\mu_0(y)
\]

\[
= \int_{\mathbb{Z}_p} \binom{\lambda y - x + n - 1}{n} d\mu_0(y)
\]

\[
= \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \binom{y\lambda - x}{m} d\mu_0(y)
\]

\[
= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{1}{m!} \int_{\mathbb{Z}_p} \binom{y\lambda - x}{m} d\mu_0(y)
\]

\[
= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{D_{m,\lambda}(-x)}{m!}.
\]

Therefore, by (37) and (38), we obtain the following theorem.

**Theorem 6.** For \( n \geq 1 \), we have

\[
(-1)^n \frac{D_{n,\lambda}(x)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{D_{m,\lambda}(-x)}{m!},
\]

and

\[
(-1)^n \frac{\hat{D}_{n,\lambda}(x)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{D_{m,\lambda}(-x)}{m!}.
\]
A note on the lambda-Daehee polynomials

Remark. For $n \geq 1$, we have

$$(-1)^n \frac{D_{n,\lambda}^{(k)}(x)}{n!}$$

$$= (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{x + \lambda x_1 + \cdots + \lambda x_k}{n} \right) d\mu_0(x_1) \cdots d\mu_0(x_k)$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( -\frac{(\lambda x_1 + \cdots + \lambda x_k) - x + n - 1}{n} \right) d\mu_0(x_1) \cdots d\mu_0(x_k)$$

$$= \sum_{m=0}^n \binom{n-1}{n-m} \frac{n}{m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( -\frac{\lambda x_1 + \cdots + \lambda x_k - x}{m} \right) d\mu_0(x_1) \cdots d\mu_0(x_k)$$

$$= \sum_{m=1}^n \binom{n-1}{m-1} \frac{1}{m!} \hat{D}_{m,\lambda}^{(k)}(x) - x$$

and

$$(-1)^n \frac{\hat{D}_{n,\lambda}^{(k)}(x)}{n!}$$

$$= (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{-\lambda x_1 + \cdots + \lambda x_k + x}{n} \right) d\mu_0(x_1) \cdots d\mu_0(x_k)$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{\lambda x_1 + \cdots + \lambda x_k - x + n - 1}{n} \right) d\mu_0(x_1) \cdots d\mu_0(x_k)$$

$$= \sum_{m=0}^n \binom{n-1}{n-m} \frac{n}{m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{\lambda x_1 + \cdots + \lambda x_k - x}{m} \right) d\mu_0(x_1) \cdots d\mu_0(x_k)$$

$$= \sum_{m=1}^n \binom{n-1}{m-1} \frac{1}{m!} \hat{D}_{m,\lambda}^{(k)}(x) - x$$

$$= \sum_{m=1}^n \binom{n-1}{m-1} \frac{1}{m!} \hat{D}_{m,\lambda}^{(k)}(x)$$

**References**


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