On the Generalized Hyers-Ulam Stability of a 4-Dimensional Quadratic-Additive Type Functional Equation

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Abstract. In this paper, we investigate the generalized Hyers-Ulam stability of a functional equation

\[
\begin{align*}
&f(x + y + z + w) + 2f(x) + 2f(y) + 2f(z) + 2f(w) - f(x + y) \\
&- f(x + z) - f(x + w) - f(y + z) - f(y + w) - f(z + w) = 0.
\end{align*}
\]

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1. Introduction

Throughout this paper, let \( X \) be a real normed space and \( Y \) a Banach space. For a given mapping \( f : X \rightarrow Y \), we define

\[
\begin{align*}
Af(x, y) &:= f(x + y) - f(x) - f(y), \\
Qf(x, y) &:= f(x + y) + f(x - y) - 2f(x) - 2f(y)
\end{align*}
\]

for all \( x, y \in X \). A mapping \( f : X \rightarrow Y \) is called an additive mapping (a quadratic mapping, respectively) if \( f \) satisfies the functional equation \( Af = 0 \) \((Qf = 0, \text{respectively})\). If a mapping is represented by sum of an additive
mapping and a quadratic mapping, we call the mapping a quadratic-additive mapping. For a functional equation \( Ef = 0 \) if all of the solutions of \( Ef = 0 \) are quadratic-additive mappings and all of quadratic-additive mappings are the solutions of \( Ef = 0 \), then we call the functional equation \( Ef = 0 \) a quadratic additive type functional equation.

S. M. Ulam [14] raised the stability problem of group homomorphisms and D. H. Hyers [4] gave a partial solution of Ulam’s problem for the case of approximate additive mappings. Hyers’ result was generalized by T. Aoki[1], Th. M. Rassias [13], and P. Găvruta[3]. During the last decades the stability problems of functional equations have been extensively investigated by a number of mathematicians [8]-[10], [12].

I.-S. Chang et. al [2] obtained the stability of the functional equation

\[
\begin{align*}
& f(x + y + z + w) + 2f(x) + 2f(y) + 2f(z) + 2f(w) - f(x + y) \\
& - f(x + z) - f(x + w) - f(y + z) - f(y + w) - f(z + w) = 0
\end{align*}
\]

(1.1)

for all \( x, y, z, w \in X \) (see also [5]-[7]). The functional equation (1.1) is a quadratic-additive type functional equation (see Theorem 2.6 in [11]). The function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = ax^2 + bx \) satisfies the functional equation (1.1), where \( a, b \) are real constants.

In this paper, we generalized the previous results of the functional equation (1.1) by adopting the different method of proof and showing the stability of the functional equation (1.1) on the punctured domain. In particular, we show the superstability(if \( p < 0 \)) of the functional equation (1.1) in the sense of Th. M. Rassias.

2. Main results

Let \((s, t)\) be a fixed element in \(((1, 1), (1, -1), (-1, -1))\) and let \( \varphi : (X \setminus \{0\})^4 \rightarrow [0, \infty) \) be a function satisfying the conditions:

\[
\begin{align*}
\sum_{j=0}^{\infty} 4^{-sj} \varphi(2^{sj} x, 2^{sj} y, 2^{sj} z, 2^{sj} w) & < \infty, \\
\sum_{j=0}^{\infty} 2^{-tj} \varphi(2^{tj} x, 2^{tj} y, 2^{tj} z, 2^{tj} w) & < \infty
\end{align*}
\]

(2.1) (2.2)

for all \( x, y, z, w \in X \setminus \{0\} \). For convenience, we use the following abbreviations for a given mapping \( f : X \rightarrow Y \):

\[
Df(x, y, z, w) := f(x + y + z + w) + 2f(x) + 2f(y) + 2f(z) + 2f(w) \\
- f(x + y) - f(x + z) - f(x + w) - f(y + z) \\
- f(y + w) - f(z + w)
\]
Lemma 2.2. (Theorem 3.1 in [11]) Let $\phi : X \backslash \{0\} \rightarrow [0, \infty)$ be a function satisfying one of the following conditions

$$
\lim_{n \to \infty} \frac{\phi(2^n x)}{2^n} = 0,
$$

$$
\lim_{n \to \infty} 2^n \phi \left( \frac{x}{2^n} \right) = 0 = \lim_{n \to \infty} \frac{\phi(2^n x)}{4^n},
$$

$$
\lim_{n \to \infty} 4^n \phi \left( \frac{x}{2^n} \right) = 0
$$

for all $x, y, z, w \in X$ and

$$
J_n f(x) = \frac{1}{2} \left( 4^{-sn} f(2^m x) + f(-2^m x) - f(0) + 2^{-tn} \left( f(2^n x) - f(-2^n x) \right) \right)
$$

for all $x \in X$. From these, we get

$$
J_n f(x) - J_{n+1} f(x) = \frac{4^{-s,n}}{2} \left( Df(2^{s,n} x, -2^{s,n} x, 2^{s,n} x, -2^{s,n} x) \right) s
$$

$$
+ 2^{r-n-2} \left( Df(2^{t,n} x, 2^{t,n} x, 2^{t,n} x, -2^{t,n} x) \right) t
$$

(2.3)

$$
-Df(-2^{t,n} x, -2^{t,n} x, -2^{t,n} x, 2^{t,n} x) t
$$

for all $x \in X$ and all nonnegative integer $n$, where $\tau_{k,n}$ are the integers defined by

$$
\tau_{k,n} = k \left( n + \frac{1}{2} \right) - \frac{1}{2}
$$

for $k \in \{-1, 1\}$.

Lemma 2.1. If $f : X \rightarrow Y$ is a mapping such that

$$
Df(x, y, z, w) = 0
$$

for all $x, y, z, w \in X \backslash \{0\}$, then $f$ is a quadratic-additive mapping.

Proof. Choose $x \in X \backslash \{0\}$, then we get

$$
f(0) = \frac{1}{3} \left( Df(x, x, x, x) + Df(-x, -x, -x, -x) + Df(2x, 2x, -2x, -2x) - 2Df(x, x, -x, -x) \right) = 0.
$$

So we easily know that

$$
Df(x, y, z, 0) = Df(2x, y, z, -x) - Df(2x, y, -x, -x) + Df(2x, y, x, -x)
$$

$$
-Df(2x, z, -x, -x) + Df(2x, z, x, -x) - 2Df(2x, x, x, -x) + Df(x, -x, -x, -x) = 0
$$

for all $x, y, z \in X \backslash \{0\}$. Using a similar method, we have $Df(x, y, 0, w) = 0$, $Df(x, 0, z, w) = 0$, $Df(0, y, z, w) = 0$, $Df(0, y, 0, 0) = 0$, $Df(x, 0, z, 0) = 0$, $Df(x, 0, w) = 0$, $Df(0, y, z, 0) = 0$, $Df(0, 0, z, w) = 0$, $Df(0, y, 0, 0) = 0$, $Df(0, 0, z, 0) = 0$, $Df(0, 0, w) = 0$, and $Df(0, 0, 0) = 0$ for all $x, y, z, w \in X \backslash \{0\}$. So we obtain $Df(x, y, z, w) = 0$ for all $x, y, z, w \in X$ and $f$ is a quadratic-additive mapping by Theorem 2.6 in [11].
for all \( x \in X \setminus \{0\} \). Let \( f : X \rightarrow Y \) be a given mapping. If there exists a quadratic-additive mapping \( F : X \rightarrow Y \) such that

\[ (2.4) \quad \| f(x) - F(x) \| \leq \phi(x) \]

for all \( x \in X \setminus \{0\} \), then \( F \) is a unique quadratic-additive mapping satisfying the inequality (2.4).

**Theorem 2.3.** Suppose that \( f : X \rightarrow Y \) is a mapping such that

\[ (2.5) \quad \| Df(x, y, z, w) \| \leq \varphi(x, y, z, w) \]

for all \( x, y, z, w \in X \setminus \{0\} \) with \( \lim_{n \to \infty} J_n f(0) = 0 \). Then there exists a unique quadratic-additive mapping \( F : X \rightarrow Y \) such that

\[ (2.6) \quad \left\| f(x) - \frac{f(0)}{2} - F(x) \right\| \leq \sum_{j=0}^{\infty} \Phi_j(x) \]

for all \( x \in X \setminus \{0\} \), where \( \Phi_j \) are the mappings defined by

\[ \Phi_j(x) = \frac{4^{\tau_{s,j}}}{2} \left( \varphi(2^{\tau_{s,j}} x, 2^{\tau_{s,j}} x, 2^{\tau_{s,j}} x, -2^{\tau_{s,j}} x) 
+ 2^{\tau_{t,j}-2} \varphi(2^{\tau_{t,j}} x, 2^{\tau_{t,j}} x, 2^{\tau_{t,j}} x, -2^{\tau_{t,j}} x) 
+ \varphi(-2^{\tau_{t,j}} x, -2^{\tau_{t,j}} x, -2^{\tau_{t,j}} x, 2^{\tau_{t,j}} x) \right) \]

for all \( x \in X \).

**Proof.** It follows from (2.3) and (2.5) that

\[ \| J_n f(x) - J_{n+m} f(x) \| \]

\[ \leq \sum_{j=n}^{n+m-1} \left\| \frac{4^{\tau_{s,j}}}{2} \left( Df(2^{\tau_{s,j}} x, -2^{\tau_{s,j}} x, 2^{\tau_{s,j}} x, -2^{\tau_{s,j}} x) s 
+ 2^{\tau_{t,j}-2} \left( Df(2^{\tau_{t,j}} x, 2^{\tau_{t,j}} x, 2^{\tau_{t,j}} x, -2^{\tau_{t,j}} x) 
- Df(-2^{\tau_{t,j}} x, -2^{\tau_{t,j}} x, -2^{\tau_{t,j}} x, 2^{\tau_{t,j}} x) \right) t \right) \right\| \]

\[ \leq \sum_{j=n}^{n+m-1} \Phi_j(x) \]

for all \( x \in X \setminus \{0\} \). From (2.1), (2.2) and (2.7), it follows that the sequence \( \{ J_n f(x) \} \) is Cauchy for all \( x \in X \setminus \{0\} \). Since \( Y \) is complete, the sequence \( \{ J_n f(x) \} \) converges. From this and \( \lim_{n \to \infty} J_n f(0) = 0 \), we can define the mapping \( F : X \rightarrow Y \) by

\[ F(x) := \lim_{n \to \infty} J_n f(x) \]

for all \( x \in X \). Moreover, letting \( n = 0 \) and passing the limit \( m \to \infty \) in (2.7), we get (2.6). Notice that \( \lim_{n \to \infty} J_n f(0) = 0 \), \( \lim_{n \to \infty} 4^{-sn} (Df(2^{sn} x, 2^{sn} y, 2^{sn} z, 2^{sn} w)) = \ldots \)
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0, and \( \lim_{n \to \infty} 2^{-tn}(DF(2^{tn}x, 2^{tn}y, 2^{tn}z, 2^{tn}w)) = 0 \) for all \( x, y, z, w \in X \setminus \{0\} \).

Hence it follows from the definition of \( F \) that

\[
DF(x, y, z, w) = \frac{1}{2} \lim_{n \to \infty} 4^{-sn}(DF(2^{sn}x, 2^{sn}y, 2^{sn}z, 2^{sn}w) + DF(-2^{sn}x, -2^{sn}y, -2^{sn}z, -2^{sn}w)) + 2^{-tn} (DF(2^{tn}x, 2^{tn}y, 2^{tn}z, 2^{tn}w) - DF(-2^{tn}x, -2^{tn}y, -2^{tn}z, -2^{tn}w))
\]

\[
= 0
\]

for all \( x, y, z, w \in X \setminus \{0\} \). By Lemma 2.1, \( DF(x, y, z, w) = 0 \) for all \( x, y, z, w \in X \). Put \( \phi(x) = \sum_{j=0}^\infty \Phi_j(x) \), then we can easily show that \( \lim_{n \to \infty} \frac{\phi(2^n x)}{4^n} = 0 \) if \( (s, t) = (1, 1) \) and \( \lim_{n \to \infty} 4^n \phi(\frac{x}{2^n}) = 0 \) if \( (s, t) = (-1, -1) \). If \( (s, t) = (1, -1) \) and \( k \) is an integer with \( n/3 < k < 2n/3 \), then

\[
\lim_{n \to \infty} \frac{\phi(2^n x)}{4^n} = \lim_{n \to \infty} \sum_{j=0}^\infty \frac{2^{-1}}{4^{n+1}} \phi(2^{j+n} x, 2^{j+n} y, 2^{j+n} z, 2^{j+n} w)
\]

\[
+ \sum_{j=0}^{k-1} \frac{2^j}{4^{n+1}} \left( \phi\left(\frac{2^n x}{2^j+1}, \frac{2^n y}{2^j+1}, \frac{2^n z}{2^j+1}, \frac{2^n w}{2^j+1}\right) + \phi\left(-\frac{2^n x}{2^j+1}, -\frac{2^n y}{2^j+1}, -\frac{2^n z}{2^j+1}, \frac{2^n w}{2^j+1}\right)\right)
\]

\[
+ \sum_{j=k}^{n-1} \frac{2^j}{4^{n+1}} \left( \phi\left(\frac{2^n x}{2^j+1}, \frac{2^n y}{2^j+1}, \frac{2^n z}{2^j+1}, \frac{2^n w}{2^j+1}\right) + \phi\left(-\frac{2^n x}{2^j+1}, -\frac{2^n y}{2^j+1}, -\frac{2^n z}{2^j+1}, \frac{2^n w}{2^j+1}\right)\right)
\]

\[
+ \sum_{j=n}^\infty \frac{2^j}{4^{n+1}} \left( \phi\left(\frac{2^n x}{2^j+1}, \frac{2^n y}{2^j+1}, \frac{2^n z}{2^j+1}, \frac{2^n w}{2^j+1}\right) + \phi\left(-\frac{2^n x}{2^j+1}, -\frac{2^n y}{2^j+1}, -\frac{2^n z}{2^j+1}, \frac{2^n w}{2^j+1}\right)\right)
\]

\[
\leq \lim_{n \to \infty} \sum_{j=n}^\infty \frac{1}{4^j} \phi\left(2^j x, 2^j y, 2^j z, 2^j w\right)
\]

\[
+ \lim_{n \to \infty} \sum_{j=n-k}^{n-1} \frac{1}{4^j} \phi\left(2^j x, 2^j y, 2^j z, -2^j x\right) + \phi\left(-2^j x, -2^j y, -2^j z, 2^j x\right)
\]

\[
+ \lim_{n \to \infty} \frac{1}{2^{k+1}} \sum_{j=0}^{n-k} \frac{1}{4^j} \phi\left(2^j x, 2^j y, 2^j z, -2^j x\right) + \phi\left(-2^j x, -2^j y, -2^j z, 2^j x\right)
\]

\[
+ \lim_{n \to \infty} \frac{1}{2^{n+3}} \sum_{j=1}^\infty 2^j \phi\left(\frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}\right) + \phi\left(-\frac{x}{2^j}, -\frac{x}{2^j}, -\frac{x}{2^j}, \frac{x}{2^j}\right)
\]

\[
= 0
\]
and

\[
\lim_{n \to \infty} 2^n \phi \left( \frac{x}{2^n} \right) = \lim_{n \to \infty} \sum_{j=0}^{\infty} 2^n \Phi_j \left( \frac{x}{2^n} \right)
\]

\[
= \lim_{n \to \infty} \left( \sum_{j=0}^{k-1} + \sum_{j=k}^{n-1} + \sum_{j=n}^{\infty} \right) \frac{2^n-1}{4^{j+1}} \phi \left( \frac{2^j x, 2^j x, 2^j x}{2^n, 2^n, 2^n}, -\frac{2^j x}{2^n} \right)
\]

\[
+ \lim_{n \to \infty} \sum_{j=0}^{\infty} \frac{2^{n+j}}{2^j} \phi \left( \frac{x}{2^{n+j+1}}, \frac{x}{2^{n+j+1}}, \frac{x}{2^{n+j+1}}, -\frac{x}{2^{n+j+1}} \right)
\]

\[
+ \lim_{n \to \infty} \sum_{j=0}^{\infty} \frac{2^{n+j}}{2^j} \phi \left( \frac{-x}{2^{n+j+1}}, \frac{-x}{2^{n+j+1}}, \frac{-x}{2^{n+j+1}}, \frac{x}{2^{n+j+1}} \right)
\]

\[
\leq \lim_{n \to \infty} \sum_{j=n-k+1}^{n} \frac{2^j}{8} \phi \left( \frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, -\frac{x}{2^j} \right)
\]

\[
+ \lim_{n \to \infty} \frac{1}{2^{k+3}} \sum_{j=1}^{n-k} \frac{2^j}{8} \phi \left( \frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, -\frac{x}{2^j} \right)
\]

\[
+ \lim_{n \to \infty} \frac{1}{2^{n+3}} \sum_{j=1}^{\infty} \frac{2^j}{8} \phi \left( \frac{2^j x, 2^j x, 2^j x}{2^j, 2^j, 2^j}, -\frac{2^j x}{2^j} \right)
\]

\[
+ \lim_{n \to \infty} \sum_{j=n+1}^{\infty} \frac{2^j}{8} \phi \left( \frac{x}{2^j}, \frac{x}{2^j}, \frac{x}{2^j}, -\frac{x}{2^j} \right)
\]

\[
+ \lim_{n \to \infty} \sum_{j=n+1}^{\infty} \frac{2^j}{8} \phi \left( \frac{-x}{2^j}, \frac{-x}{2^j}, \frac{-x}{2^j}, \frac{x}{2^j} \right)
\]

\[
= 0
\]

for all \(x \in X\backslash\{0\}\). By Lemma 2.2, \(F\) is the unique quadratic-additive mapping satisfying (2.6).

\textbf{Corollary 2.4.} Let \(p \neq 1, 2\) be a real number. Suppose that \(f : X \to Y\) is a mapping satisfying

(2.8) \[\|Df(x, y, z, w)\| \leq \|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p\]

for all \(x, y, z, w \in X\backslash\{0\}\) (with \(f(0) = 0\) if \(p > 2\)). Then there exists a unique quadratic-additive mapping \(F\) such that

(2.9) \[\left\|f(x) - \frac{f(0)}{2} - F(x)\right\| \leq \left(\frac{2}{|2^p - 4|} + \frac{2}{|2^p - 2|}\right) \|x\|^p\]

for all \(x \in X\backslash\{0\}\).
Corollary 2.5. Let \( p < 0 \) be a real number. Suppose that \( f : X \to Y \) is a mapping satisfying (2.8) for all \( x, y, z, w \in X \{ 0 \} \). Then \( f \) satisfies \( Df(x, y, z, w) = 0 \) for all \( x, y, z, w \in X \).

Proof. By Corollary 2.4, there exists a mapping \( F(x) \) satisfying \( DF(x, y, z, w) = 0 \) for all \( x, y, z, w \in X \) and (2.9) for all \( x \in X \{ 0 \} \). From these, we get \( F(0) = 0 \) and

\[
3\|f(x) - F(x) - f(0)\| \\
\leq \|(Df - DF)((n + 1)x, -nx, -nx, -nx)\| \\
+ \left\| (f - F)((-2n + 1)x) - \frac{f(0)}{2} \right\| + 3\|f - F(-2nx) - \frac{f(0)}{2}\| \\
+ 6\left\| (f - F)(-nx) - \frac{f(0)}{2} \right\| + 2\|f - F((n + 1)x) - \frac{f(0)}{2}\| \\
\leq \left( \frac{2}{|2p - 4|} + \frac{2}{|2p - 2|} \right) (2(n + 1)^p + (2n - 1)^p + 6n^p + 3(2n)^p) \|x\|^p \\
+ ((n + 1)^p + 3n^p)\|x\|^p
\]

for all \( x \in X \{ 0 \} \) and all positive integer \( n \). Taking the limit as \( n \to \infty \) in the above inequality, we get \( f(x) = f(x) - f(0) \) for all \( x \in X \{ 0 \} \). By (2.9) and this, we have

\[
\left\| \frac{f(0)}{2} \right\| = \left\| f(x) - \frac{f(0)}{2} - F(x) \right\| \leq \left( \frac{2}{|2p - 4|} + \frac{2}{|2p - 2|} \right) \|x\|^p
\]

for all \( x \in X \{ 0 \} \). Hence \( f(0) = 0 \) and \( f(x) = F(x) \) for all \( x \in X \).

\( \square \)

Lemma 2.6. If \( f : X \to Y \) is a mapping such that

\[
Df(x, y, z, w) = 0
\]

for all \( x, y, z, w \in X \) and \( f(tx) \) is continuous in \( t \) for each fixed \( x \). Then \( f \) is represented by

\[
f(rx) = \left( \frac{f(x) + f(-x)}{2} \right) r^2 + \left( \frac{f(x) - f(-x)}{2} \right) r
\]

for all \( x \in X \) and all \( r \in \mathbb{R} \).

Proof. First claim that

\[
f(nx) = \left( \frac{f(x) + f(-x)}{2} \right) n^2 + \left( \frac{f(x) - f(-x)}{2} \right) n
\]
for all \( x \in X \) and all nonnegative integer \( n \). Note that \( f(0) = \frac{1}{2}Df(0, 0, 0, 0) = 0 \). For \( n = 0, 1 \), it is trivial. For \( n = 2 \), we can show that

\[
\begin{align*}
\frac{f(2x)}{2} &= -Df(x, x, -x, 0) + 3f(x) + f(-x) \\
&= \left(\frac{f(x) + f(-x)}{2}\right)2^2 + \left(\frac{f(x) - f(-x)}{2}\right)2
\end{align*}
\]

for all \( x \in X \). Assume that (2.10) holds for all \( x \in X \) and all nonnegative integer \( k(\leq n) \). Then

\[
\begin{align*}
f((n+1)x) &= -Df(nx, x, -x, 0) - f((n-1)x) + 2f(nx) + f(x) + f(-x) \\
&= \left(\frac{f(x) + f(-x)}{2}\right)(2n^2 - (n-1)^2 + 2) + \left(\frac{f(x) - f(-x)}{2}\right)(n+1)
\end{align*}
\]

for all \( x \in X \). From (2.10), we get

\[
\begin{align*}
\frac{f(nx) + f(-nx)}{2} &= \left(\frac{f(x) + f(-x)}{2}\right)n^2, \\
\frac{f(nx) - f(-nx)}{2} &= \left(\frac{f(x) - f(-x)}{2}\right)n, \\
\frac{f\left(\frac{x}{n}\right) + f\left(-\frac{x}{n}\right)}{2} &= \left(\frac{f(x) + f(-x)}{2}\right)\frac{1}{n^2}, \\
\frac{f\left(\frac{x}{n}\right) - f\left(-\frac{x}{n}\right)}{2} &= \frac{f(x) - f(-x)}{2n}
\end{align*}
\]

for all \( x \in X \) and all integer \( n \neq 0 \). Hence

\[
\begin{align*}
f\left(\frac{p}{q}x\right) &= \frac{f\left(\frac{p}{q}x\right) + f\left(-\frac{p}{q}x\right)}{2} + \frac{f\left(\frac{p}{q}x\right) - f\left(-\frac{p}{q}x\right)}{2} \\
&= \left(\frac{f\left(\frac{x}{q}\right) + f\left(-\frac{x}{q}\right)}{2}\right)p^2 + \left(\frac{f\left(\frac{x}{q}\right) - f\left(-\frac{x}{q}\right)}{2}\right)p \\
&= \left(\frac{f(x) + f(-x)}{2}\right)p^2 + \left(\frac{f(x) - f(-x)}{2}\right)p
\end{align*}
\]

for all \( x \in X \) and all integer \( p, q(\neq 0) \). For \( r \in \mathbb{R} \), let \( \{r_n\} \) be a rational sequence satisfying \( \lim_{n \to \infty} r_n = r \). Since \( f(tx) \) is continuous in \( t \) for each fixed \( x \), we have

\[
\begin{align*}
f(rx) &= \lim_{n \to \infty} f(r_n x) \\
&= \lim_{n \to \infty} \left(\frac{f(x) + f(-x)}{2}\right)r_n^2 + \left(\frac{f(x) - f(-x)}{2}\right)r_n \\
&= \left(\frac{f(x) + f(-x)}{2}\right)r + \left(\frac{f(x) - f(-x)}{2}\right)r
\end{align*}
\]
for all $x \in X$.

The following corollary follows from Corollary 2.5 and Lemma 2.6.

**Corollary 2.7.** Let $p < 0$ be a real number. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a mapping satisfying

$$|Df(x, y, z, w)| \leq |x|^p + |y|^p + |z|^p + |w|^p$$

for all $x, y, z, w \in \mathbb{R}\setminus\{0\}$ and $f$ is continuous. Then $f$ is represented by

$$f(x) = \left(\frac{f(1) + f(-1)}{2}\right)x^2 + \left(\frac{f(1) - f(-1)}{2}\right)x$$

for all $x \in \mathbb{R}$.

**Corollary 2.8.** Let $\varphi : X^4 \to [0, \infty)$ be a function satisfying the conditions (2.1) and (2.2) for all $x, y, z, w \in X$. Suppose that $f : X \to Y$ is a mapping satisfying (2.5) for all $x, y, z, w \in X$. Then there exists a unique quadratic-additive mapping mapping $F$ satisfying (2.6) for all $x \in X$, where $\Phi_j$ are as in Theorem 2.3.

**Proof.** By the definition of $J_n$, we have

$$\|J_nf(0)\| = \left\|\frac{4^{-sn}}{2}f(0)\right\|$$

$$= \left\|\frac{4^{-sn}}{6}Df(2^{sn}0, 2^{sn}0, 2^{sn}0, 2^{sn}0)\right\|$$

$$\leq \frac{4^{-sn}}{6}\varphi(2^{sn}0, 2^{sn}0, 2^{sn}0, 2^{sn}0).$$

From (2.1), we get $\lim_{n \to \infty} J_nf(0) = 0$ and this theorem follows from Theorem 2.3. □

**References**


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