Some New Fixed Point Theorems
in 2-Normed Spaces

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Abstract. In this paper, we introduce and prove some new fixed point theorems in 2-normed spaces.

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1. Introduction and Preliminaries

Fixed point theory is one of the most important topics in development of the functional analysis. Also, it has been used effectively in many branch of science, such as chemistry, biology, economics, computer science, engineering and so on.

Recently, Harikrishnan and Ravindran gave some new properties of accretive mappings and contraction mappings in 2-normed spaces [3].

In the fixed point theory there are many different iteration scheme. In this study we use Picard iteration given as following:

For any $x_0 \in X$, the sequence $\{x_n\}_{n \geq 0} \subset X$ given by

$$x_n = Tx_{n-1} = T^nx_0, \ n = 1, 2, ...$$

(1.1)
is called the sequence of successive approximations with the initial value $x_0$. It is also known as the Picard iteration starting at $x_0$ [2].

A mapping $T : X \to X$ where $(X, d)$ is a metric space, is said to be a contraction if there exists $k \in [0, 1)$ such that for all $x, y \in X$,

\begin{equation}
    d(Tx, Ty) \leq kd(x, y).
\end{equation}

If the metric space $(X, d)$ is complete then the mapping satisfying (1.2) has a unique fixed point [4].

Also, Kannan [5] established the following result:

If a mapping $T : X \to X$ where $(X, d)$ is a complete metric space, satisfies the inequality

\begin{equation}
    d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]
\end{equation}

where $a \in [0, \frac{1}{2})$ and $x, y \in X$. Then, $T$ has a unique fixed point. The mappings satisfying (1.3) are called Kannan type mappings.

A similar contractive condition has been introduced by Chatterjea [6] as following:

If $T : X \to X$ where $(X, d)$ is a complete metric space, satisfies the inequality

\begin{equation}
    d(Tx, Ty) \leq b[d(x, Ty) + d(y, Tx)]
\end{equation}

where $b \in [0, \frac{1}{2})$ and $x, y \in X$. Then, $T$ has a unique fixed point. The mappings satisfying (1.4) are called Chatterjea type mapping.

Now, we give some basic definitions and results in 2-normed spaces.

**Definition 1.** [1] Let $E$ be a real linear space with $\dim E \geq 2$ and $\|.,.\| : E \times E \to [0, \infty)$ be a function. Then, $(E, \|.,.\|)$ is called a linear $2 -$ normed space for all $x, y, z \in E$ and $\alpha \in \mathbb{R}$,

\begin{align*}
    (2N_1) & \quad \|x, y\| = 0 \text{ if and only if } x \text{ and } y \text{ are linearly dependent}, \\
    (2N_2) & \quad \|x, y\| = \|y, x\|, \\
    (2N_3) & \quad \|\alpha x, y\| = \|\alpha\| \|x, y\|, \\
    (2N_4) & \quad \|x + y, z\| \leq \|x, z\| + \|y, z\|.
\end{align*}

**Definition 2.** [1] A sequence $\{x_n\}$ in a 2-normed space $(E, \|.,.\|)$ is said to be a Cauchy sequence if $\lim_{m,n \to \infty} \|x_n - x_m, z\| = 0$ for all $z \in E$.

**Definition 3.** [1] A sequence $\{x_n\}$ in a 2-normed space $(E, \|.,.\|)$ is said to be convergent if there is a point $x$ in $E$ such that $\lim_{n \to \infty} \|x_n - x, z\| = 0$ for all $z$ in $E$. If $\{x_n\}$ converges to $x$ we write $x_n \to x$ as $n \to \infty$.

**Definition 4.** [1] A linear 2-normed space is said to be complete if every Cauchy sequence is convergent to an element of $E$. A complete 2-normed space $E$ is called 2-Banach space.
Definition 5. [3] Let \((E, \|\cdot\|)\) be a linear 2-normed space, \(C\) be a subset of \(E\) then the closure of \(C\) is \(\overline{C} = \{x \in E; \text{there is a sequence } x_n \text{ of } C \text{ such that } x_n \to x\}\). We say, \(C\) is sequentially closed if \(C = \overline{C}\).

Definition 6. [3] Let \((E, \|\cdot\|)\) be a linear 2-normed space, \(B\) be a nonempty subset of \(E\) and \(e \in B\) then \(B\) is said to be \(e\)-bounded if there exist some \(M > 0\) such that \(\|x, e\| \leq M\) for all \(x \in B\). If for all \(e \in B\), \(B\) is \(e\)-bounded then \(B\) is called a bounded set.

Harikrishnan and Ravindran [3] introduced contraction mappings in 2-normed spaces as following:

Definition 7. Let \((E, \|\cdot\|)\) be a linear 2-normed space. Then the mapping \(T : E \to E\) is said to be a contraction if there exists \(k \in [0, 1)\) such that
\[
\|Tx - Ty, z\| \leq k \|x - y, z\| \quad (1.5)
\]
for all \(x, y, z \in X\).

Harikrishnan and Ravindran [3] proved that a contraction mapping has a unique fixed point in closed and bounded subset of 2-normed spaces.

2. Main Results

In this section, we give Kannan fixed point theorem and Chatterjea fixed point theorem arising from 2-normed spaces.

Theorem 1. Let \((E, \|\cdot\|)\) be a linear 2-Banach space and \(K\) be a nonempty closed and bounded subset of \(E\). If \(T : K \to K\) satisfying
\[
\|Tx - Ty, z\| \leq \alpha \left( \|x, x\| + \|y, y\| \right) \quad (2.1)
\]
where \(\alpha \in \left[0, \frac{1}{2}\right]\) then, \(T\) has unique fixed point in \(K\).

Proof. Let \(x_0 \in X\) and \(\{x_n\}_{n=1}^{\infty}\) be a sequence in \(K\) defined by Picard iteration scheme. We have
\[
\|x_n - x_{n+1}, z\| = \|Tx_{n-1} - Tx_n, z\| \leq \alpha \left( \|x_{n-1} - x_n, z\| + \|x_n - x_{n+1}, z\| \right) \quad (2.2)
\]
and we obtain
\[
\|x_n - x_{n+1}, z\| \leq \frac{\alpha}{1 - \alpha} \|x_n - x_{n+1}, z\|. \quad (2.3)
\]
Note that \(\alpha \in \left[0, \frac{1}{2}\right]\), then \(k := \frac{\alpha}{1 - \alpha} \in [0, 1)\). Thus, \(T\) is a contraction mapping. Also, from (2.3) we have
\[
\|x_n - x_{n+1}, z\| \leq \left( \frac{\alpha}{1 - \alpha} \right)^n \|x_0 - x_1, z\|. 
\]
Now, we show that \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence. Let \( m, n > 0 \) with \( m > n \).
Taking \( m = n + p \)
\[
\|x_n - x_m, z\| = \|x_n - x_{n+p}, z\| \\
\leq \|x_n - x_{n+1}, z\| + \|x_{n+2} - x_{n+1}, z\| \\
\cdots + \|x_{n+p-1} - x_{n+p}, z\|.
\]
(2.4)
From (2.4), we have
\[
\|x_n - x_m, z\| \leq k^n \|x_0 - x_1, z\| + k^{n+1} \|x_0 - x_1, z\| \\
\cdots + k^{n+p-1} \|x_0 - x_1, z\|.
\]
(2.5)
Letting \( m, n \to \infty \) in (2.5), we have
\[
\lim_{n \to \infty} \|x_n - x_m, z\| = \lim_{n \to \infty} \|x_n - x_{n+p}, z\| \\
\leq 0.
\]
(2.6)
Thus, \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( K \). Hence, there exists \( y \in K \) such that
\[
\lim_{n \to \infty} x_n = y.
\]
(2.7)
Now, we show that \( y \in K \) is a fixed point of \( T \). Using triangle inequality of 2-norm, we arrive at
\[
\|Ty - y, z\| \leq \|Ty - x_n, z\| + \|x_n - y, z\|.
\]
(2.8)
Letting \( n \to \infty \) in (2.8), we have
\[
\lim_{n \to \infty} \|Ty - y, z\| = 0.
\]
Therefore, \( y = Ty \) and implies that \( y \) is a fixed point of \( T \).
Now, assume that \( y' \) is another fixed point of \( T \). Thus, we have \( Ty = y' \) and
\[
\|y - y', z\| \leq \|Ty - Ty', z\| \\
\leq \alpha [\|y - Ty, z\| + \|y' - Ty', z\|] \\
= 0.
\]
(2.9)
The inequality (2.9) implies that \( y = y' \). Hence, we obtain that the fixed point is unique. This completes the proof. \( \square \)

**Theorem 2.** Let \( (E, \|\cdot, \cdot\|) \) be a linear 2-Banach space and \( K \) be a nonempty closed and bounded subset of \( E \). If \( T : K \to K \) satisfying
\[
\|Tx - Ty, z\| \leq \beta [\|x - Ty, z\| + \|y - Tx, z\|]
\]
where \( \beta \in [\frac{1}{2}, 1) \) then, \( T \) has unique fixed point in \( K \).
Proof. Let \( x_0 \in X \) and \( \{x_n\}_{n=1}^{\infty} \) be a sequence in \( X \) defined as \( x_n = T x_{n-1} = T^n x_0, \ n = 1, 2, \ldots \). We have

\[
\|x_n - x_{n+1}, z\| = \|T x_{n-1} - T x_n, z\| \\
\leq \beta \left( \|x_{n-1} - T x_n, z\| + \|x_n - T x_{n-1}, z\| \right) \\
= \beta \|x_{n-1} - x_n, z\| \\
\leq \beta \left( \|x_{n-1} - x_n, z\| + \|x_n - x_{n+1}, z\| \right)
\]

From (2.11), we obtain that

\[
\|x_n - x_{n+1}, z\| \leq \frac{\beta}{1-\beta} \|x_{n-1} - x_n, z\|
\]

Note that \( \beta \in \left[0, \frac{1}{2}\right] \), then \( k := \frac{\beta}{1-\beta} \in (0, 1) \). Also from (2.12), we have

\[
\|x_n - x_{n+1}, z\| \leq k^n \|x_0 - x_1, z\|.
\]

Using the triangle inequality of 2-norm, we have

\[
\|x_n - x_m, z\| \leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{m-2} - x_{m-1}, z\| \\
\leq k^n \|x_0 - x_1, z\| \left[ 1 + k + k^2 + \cdots \right].
\]

Letting \( m, n \to \infty \) in (2.14), we have

\[
\lim_{m,n \to \infty} \|x_n - x_m, z\| = 0.
\]

Thus, we obtain that \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( K \). Hence there exists \( x^* \in K \) such that \( (x_n) \to x^* \) as \( n \to \infty \).

Now, we will show that \( x^* \in K \) is a fixed point of \( T \).

\[
\|x^* - T x^*, z\| \leq \|x^* - x_n, z\| + \|x_n - T x^*, z\| \\
\leq \|x^* - x_n, z\| + \|x_n - x_{n+1}, z\| + \|x_{n+1} - T x^*, z\| \\
\leq \|x^* - x_n, z\| + k^n \|x_0 - x_1, z\| + \|T x_n - T x^*, z\| \\
\leq \|x^* - x_n, z\| + k^n \|x_0 - x_1, z\| \\
+ \beta \left( \|x_n - T x^*, z\| + \|x^* - T x_n, z\| \right).
\]

Letting \( n \to \infty \) in (2.16), we obtain that

\[
\|x^* - T x^*, z\| \leq 2\beta \|x^* - T x^*, z\|.
\]

Inequality (2.17) is contradiction unless \( \|x^* - T x^*, z\| = 0 \). This implies that \( x^* = T x^* \) and we obtain that \( x^* \in K \) is a fixed point of \( T \). To complete the proof we now show that the fixed point is unique. Assume that \( x' \in K \) is another fixed point of \( T \). Hence, we have \( T x' = x' \) and

\[
\|x^* - x', z\| = \|T x^* - T x', z\| \\
\leq \beta \left( \|x^* - T x', z\| + \|x' - T x^*, z\| \right) \\
= 2\beta \|x^* - T x', z\|.
\]

(2.18)
The inequality (2.18) is contradiction with $\beta \in [0, \frac{1}{2})$ unless $\|x^* - x', z\| = 0$. Thus, $x^* = x'$. This completes the proof.

\[\square\]

REFERENCES


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