Existence and Uniqueness of Large Solutions for Competitive Type Elliptic Systems

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Abstract

In this paper, we establish the results concerning existence, non-existence, uniqueness and nonuniqueness of large solutions to competitive type elliptic systems.

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1 Introduction

In this paper, we consider the existence and uniqueness of solutions for the following elliptic system

$$
\begin{align*}
\Delta u &= e^{mu+pv}, \quad \Delta v = e^{qu+nv} \quad \text{in } \Omega, \\
 u &= v = +\infty \quad \text{on } \partial \Omega,
\end{align*}
$$

(1.1)

where $m, n, p, q > 0$, $mn \geq pq$ and $\Omega$ is a bounded $C^2$ domain of $R^N$. The boundary condition is to be understood as $u(x) \to +\infty$, $v(x) \to +\infty$ as $d(x) \to 0+$, where $d(x)$ stands for the distance function $\text{dist}(x, \partial \Omega)$. Problems like (1.1) are usually known in the literature as boundary blow-up problems, and their solutions are also named large solutions or boundary blow-up solutions.
There is a vast literature on elliptic problems that have solutions which blow up. Starting with the pioneering papers [1], problems related to large solutions have a long history, arise naturally from a number of different areas and are studied by many authors and in many contexts.

Large solutions of the elliptic problems

\[
\begin{align*}
\Delta u &= f(x, u) & \text{in } \Omega, \\
u &= +\infty & \text{on } \partial\Omega,
\end{align*}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) \((N \geq 1)\) have been extensively studied, see [1-4, 6-8]. In 1916, Bieberbach [1] studied (1.2) with \(f(x, u) = e^u\), and showed that if \(\Omega\) is a bounded domain in \(\mathbb{R}^2\) such that \(\partial\Omega\) is a \(C^2\) submanifold of \(\mathbb{R}^2\), then there exists a unique \(u \in C^2(\Omega)\) satisfying (1.2) and \(|u(x) + 2 \log d(x)|\) is bounded on \(\Omega\). Lazer and McKenna [8] extended the results for a bounded domain \(\Omega\) in \(\mathbb{R}^N\) \((N \geq 1)\) satisfying a uniform external sphere condition and the non-linearity \(f(x, u) = a(x)e^u\), where \(a(x)\) is continuous and strictly positive on \(\overline{\Omega}\). Recently, Chuaqui et al. [2] considered the (1.2) with \(f(x, u) = a(x)u^p\) or \(f(x, u) = a(x)e^u\), here \(a(x)\) can be unbounded, which satisfies \(C_1d(x)^{-\lambda} \leq a(x) \leq C_2d^{-\lambda}(x)\) in \(\Omega\) for some \(C_1, C_2 > 0\) and \(\lambda \in (0, 2)\).

Recently, large solutions of the elliptic systems have received much attention. We quote [5,9] and reference therein for competitive type systems. However in these papers, they only considered the elliptic systems coupled via power nonlinearities. The aim of this paper is to prove the existence and uniqueness of large solutions for elliptic systems (1.1).

Now we state the main results of this paper.

**Theorem 1.1** Under the assumption \(mn > pq\), the problem (1.1) has a solution \((u, v)\) if and only if \(m > q, n > p\). Moreover, this solution is unique.

**Theorem 1.2** Under the assumption \(mn = pq\), the problem (1.1) has a solution \((u, v)\) if and only if \(m = q, n = p\). Moreover, If \((u, v)\) is a solution to (1.1), then \((u + \log \epsilon, v - \frac{m}{n} \log \epsilon)\) is also a solution for every \(\epsilon > 0\), and thus there are infinitely many solutions.

This paper is organized as follows. In the next section, we give some preliminaries which will be used in the proofs of Theorems 1.1 and 1.2. In Section 3, we give the proofs of Theorems 1.1 and 1.2.

### 2 Preliminaries

We give the definition of a subsolution and a supersolution to (1.1).
Definition 2.1 \((\underline{u}, \underline{v})\) is a subsolution to (1.1) if
\[
\Delta \underline{u} \geq e^{\underline{m}u + p\underline{v}}, \quad \Delta \underline{v} \leq e^{\underline{q}u + n\underline{v}} \quad \text{in } \Omega,
\]
\[
\underline{u} = \underline{v} = +\infty \quad \text{on } \partial \Omega.
\] (2.1)

A supersolution \((\overline{u}, \overline{v})\) is defined by reversing the inequalities in (2.1).

Following the Chapter 3 in the book [10], we have the following result.

Lemma 2.2 Let \((\underline{u}, \underline{v})\) and \((\overline{u}, \overline{v})\) satisfy \(\underline{u} \leq \overline{u}, \underline{v} \geq \overline{v}\) in \(\Omega\) and
\[
\Delta \underline{u} \geq e^{\underline{m}u + p\underline{v}}, \quad \Delta \underline{v} \leq e^{\underline{q}u + n\underline{v}}, \quad \Delta \overline{u} \leq e^{\overline{m}u + p\overline{v}}, \quad \Delta \overline{v} \geq e^{\overline{q}u + n\overline{v}} \quad \text{in } \Omega,
\]
\[
\underline{u} \leq f(x) \leq \overline{u}, \quad \underline{v} \geq g(x) \geq \overline{v} \quad \text{on } \partial \Omega.
\]

Then the problem
\[
\Delta u = e^{\underline{m}u + p\overline{v}}, \quad \Delta \overline{v} = e^{\overline{q}u + n\underline{v}} \quad \text{in } \Omega,
\]
\[
u = f(x), \quad v = g(x) \quad \text{on } \partial \Omega,
\]
has at least a solution \((u, v)\) with \(\underline{u} \leq u \leq \overline{u}, \underline{v} \geq v \geq \overline{v}\) in \(\Omega\), where \(f, g\) are continuous functions defined on \(\partial \Omega\).

In order to prove the existence of solutions to (1.1), we give the following lemma, the proof of which can be found in [3].

Lemma 2.3 Assume \((\underline{u}, \underline{v})\) and \((\overline{u}, \overline{v})\) are a subsolution and a supersolution of (1.1) with \(\underline{u} \leq \overline{u}, \underline{v} \geq \overline{v}\) in \(\Omega\). Then the problem (1.1) has at least a solution 
\((u, v)\) with \(\underline{u} \leq u \leq \overline{u}, \underline{v} \geq v \geq \overline{v}\).

Next, we are ready to consider the single equation
\[
\Delta u = d\lambda u e^{mu} \quad \text{in } \Omega,
\]
\[
u = +\infty \quad \text{on } \partial \Omega.
\] (2.2)

The following lemma, part of Theorem 1.2 in [2], contains the basic features of the problem (2.2).

Lemma 2.4 If \(0 < \lambda < 2\), then the problem (2.2) has a unique solution denoted by \(u_{m, \lambda}\). Moreover, \(\lim_{x \to x_0} (u_{m, \lambda}(x) + \rho \log d(x)) = \frac{1}{m} \log \rho\) for every \(x_0 \in \partial \Omega\), where \(\rho = \frac{2 - \lambda}{m}\).

Let \(z = -\frac{1}{m} \log C + u_{m, \lambda}\), we have \(\Delta z = Cd^{-\lambda}(x)e^{mz}\) for \(C > 0\). Then we obtain the conclusion:

Lemma 2.5 Let \(u \in C^2(\Omega)\) satisfies \(\Delta u \leq Cd^{-\lambda}(x)e^{mu}\) (resp. \(\Delta u \geq Cd^{-\lambda}(x)e^{mu}\)) in \(\Omega\) for some positive constant \(C\) and \(u = +\infty\) on \(\partial \Omega\), then
\[
u(x) \geq -\frac{1}{m} \log C + u_{m, \lambda} \quad (\text{resp. } u(x) \leq -\frac{1}{m} \log C + u_{m, \lambda}) \quad \text{in } \Omega.
\]

Define \(L_{m, \lambda} = \sup_{x \in \Omega} (u_{m, \lambda}(x) + \rho \log d(x))\), \(l_{m, \lambda} = \inf_{x \in \Omega} (u_{m, \lambda}(x) + \rho \log d(x))\).

Following the Lemma 5 in [3], we have the following result.

Lemma 2.6 The quantities \(L_{m, \lambda}\) and \(l_{m, \lambda}\) are bounded when \(\lambda\) bounded away from 2. Moreover, \(\lim_{\lambda \to 2} L_{m, \lambda} = \lim_{\lambda \to 2} l_{m, \lambda} = -\infty\).
3 Existence and uniqueness

Lemma 3.1 When $mn > pq$, the problem (1.1) has at least a solution if $m > q, n > p$.

Proof. Let $\lambda_1 = \frac{2p(m-q)}{mn-pq}$, $\lambda_2 = \frac{2q(n-p)}{mn-pq}$. It is easy to check that $0 < \lambda_1, \lambda_2 < 2$ since $m > q, n > p$. We look for a subsolution of the form $(u, v) = (u_m, \lambda_1 + \log b, u_n, \lambda_2 - \gamma \log b)$, where the functions $u_m, \lambda_1, u_n, \lambda_2$ are as introduced in Lemma 2.4 and $b, \gamma > 0$ are to be chosen. Then $(u, v)$ is a subsolution if

$$b^m - \gamma d^{\lambda_1}(x)e^{\mu n, \lambda_2} \leq 1, \quad b^{n-\gamma} d^{\lambda_2}(x)e^{\mu n, \lambda_1} \geq 1. \quad (3.1)$$

Noting that $\frac{\lambda_1}{p} = \frac{2-\lambda_2}{n}$, $\frac{\lambda_2}{q} = \frac{2-\lambda_1}{n}$, by Lemma 2.6 we have that $d^{\lambda_1}(x)e^{\mu n, \lambda_2}, d^{\lambda_2}(x)e^{\mu n, \lambda_1}$ are bounded and bounded away from zero. Then inequalities (3.1) hold for $b$ small if $m - p\gamma > 0, q - n\gamma < 0$. Thus fixing $\frac{2}{n} < \gamma < \frac{2}{m}$, we obtain the subsolution of (1.1).

By the similar argument, we easily check that $(\bar{u}, \bar{v}) = (u_m, \lambda_1 + \log B, u_n, \lambda_2 - \gamma \log B)$ is a supersolution to (1.1) if $B > 0$ is large enough.

Since the sub- and supersolution are ordered, that is $u \leq \bar{u}, \quad v \geq \bar{v}$, Lemma 2.3 implies the existence of a solution $(u, v)$ to (1.1). $\square$

Lemma 3.2 If $mn > pq$ and $m \leq q$ or $n \leq p$, then the problem (1.1) has no solution.

Proof. Consider first $p > n$. Here, we are using a new iteration method. Let $a_0 = \inf_{x \in \Omega} e^{\nu}$, then since $\Delta u \geq a_0^p e^{\mu u}$, by Lemma 2.5, we have $u \leq -\frac{p}{m} \log a_0 + u_{m,0}$, and thus $u \leq -\frac{2}{m} \log a_0 + L_{m,0} - a_0 \log d(x)$, where $a_0 = \frac{2}{m}$. Inserting this into the second equation we have $\Delta v = e^{\mu u + n v} \leq a_0^p \frac{2}{m} e^{q L_{m,0}d^{-\alpha_0 q}(x)} e^{n v}$ and Lemma 2.5 again gives $v \geq \log(a_0^2 \frac{2}{n} e^{-\frac{2}{n} L_{m,0}} + l_{n,0} - \beta_0 \log d(x)$, where $\beta_0 = \frac{2-\alpha_0 q}{n}$. Proceeding inductively, we obtain

$$u \leq -\frac{p}{m} \log a_0 + L_{m,\beta_{n-1}} - \alpha_n \log d(x), \quad v \geq \log a_{n+1} - \beta_n \log d(x), \quad (3.2)$$

where

$$\alpha_n = \frac{2-\beta_{n-1} p}{m}, \quad \beta_n = \frac{2-\alpha_n q}{n}, \quad a_{n+1} = a_n^p e^{-\frac{2}{n} L_{m,\beta_{n-1} p + l_{n,0} - \alpha_n q}}. \quad (3.3)$$

Let us see that all these quantities converge as $n \to +\infty$. By a direct computation we have that $\beta_n = \frac{2(m-q)}{mn} + \frac{pq}{mn} \beta_{n-1}$ and $\beta_1 > \beta_0, \beta_n \leq \beta$. In particular, we deduce that $\beta_n$ converges to $\beta = \frac{2(m-q)}{mn-pq}$ as $n \to +\infty$. As a consequence, also $\alpha_n \to \alpha = \frac{2(n-p)}{mn-pq}$. Since $\alpha_0 > 0$, we can choose $n$ so that $\alpha_n > 0, \alpha_{n+1} < 0$. Thus, $\beta_{n-1} p < 2$ and $\beta_n p > 2$. Also, recall from (3.2)
that \( v \geq \log a_{n+1} - \beta_n \log d(x) \) in \( \Omega \), and thus \( \Delta u \geq a_{n+1}^p d^{-\beta_n p}e^{mu} \) in \( \Omega \).

According to Theorem 1.2 in [2], this implies that \( u \) is bounded. Thus the problem (1.1) has no solution.

Now assume \( n = p \). The iteration argument in the case \( p > n \) makes full sense, but \( \alpha = 0 \). Also, thanks to (3.2) and (3.3), we obtain \( v \geq \log a_{n+1} - \beta_n \log d(x) \) in \( \Omega \), where \( a_{n+1} = a_n^k e^{\frac{p}{m} L_{m, \beta_{n-1} p} + \log a_{n+1}} \), and \( k = \frac{pq}{mn} < 1 \). But \( p \beta_{n-1} \rightarrow 2 \) as \( n \rightarrow +\infty \), and so Lemma 2.6 implies \( L_{m, \beta_{n-1} p} \rightarrow -\infty \), while \( l_{n, qn} \) is bounded. In particular, for every \( M > 0 \) there exists \( n_0 \) such that \( a_{n+1} \geq Ma_n^k \) for \( n \geq n_0 \). This readily gives \( \liminf_{n \rightarrow +\infty} a_{n+1} > M^{\frac{1}{mp}} \), and since \( M \) is arbitrary, \( \liminf_{n \rightarrow +\infty} a_{n+1} = +\infty \). But then \( v = +\infty \) in \( \Omega \), which is not possible, and no solution exists in this case. \( \square \)

**Lemma 3.3** Let \((u_1, v_1), (u_2, v_2)\) be solutions to (1.1) such that \((u_1, v_1) = (u_2, v_2)\) on \( \partial \Omega \). If \( mn \geq pq \), then the solution to (1.1) is unique.

Proof. Define \( \Omega_\delta := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \} \). Now consider the following system

\[
\begin{align*}
\Delta u &= e^{mu+pv}, & \Delta v &= e^{qu+nv} & \text{in } \Omega_\delta, \\
u &= u_1, & v &= v_1 & \text{on } \partial \Omega_\delta.
\end{align*}
\] (3.4)

Note \((u_1, v_1) = (u_2, v_2)\) on \( \partial \Omega \). For \( \epsilon \in (0, 1) \), there exists \( \delta_0 > 0 \) such that, for \( \delta \in (0, \delta_0) \), \((u_2 + \log(1+\epsilon) - \frac{p}{m} \log(1+\epsilon), v_2 + \log(1+\epsilon) - \frac{p}{n} \log(1+\epsilon))\) and \((u_2 + \log(1+\epsilon) - \frac{p}{m} \log(1-\epsilon), v_2 + \log(1+\epsilon) - \frac{p}{n} \log(1+\epsilon))\) are a subsolution and a supersolution to (3.4). It follows from Lemma 2.2 that (3.4) has a solution \((u_\delta, v_\delta)\) such that

\[
\begin{align*}
u_2 + \log(1+\epsilon) - \frac{p}{m} \log(1+\epsilon) \leq u_\delta & \leq u_2 + \log(1+\epsilon) - \frac{p}{m} \log(1-\epsilon), \\
u_2 + \log(1+\epsilon) - \frac{p}{n} \log(1+\epsilon) \geq u_\delta & \geq v_2 + \log(1+\epsilon) - \frac{p}{n} \log(1+\epsilon).
\end{align*}
\] (3.5)

By the uniqueness of the solution to (3.4), we have

\[
u_\delta = u_1|_{\Omega_\delta}, \quad v_\delta = v_1|_{\Omega_\delta}.
\] (3.6)

Letting \( \epsilon \rightarrow 0 \) and combing (3.5),(3.6), we have \((u_1, v_1) = (u_2, v_2)\) in \( \Omega \). \( \square \)

We get the proof of Theorem 1.1 by combining Lemmas 3.1-3.3.

**Proof of Theorem 1.2.** Let us begin by proving that \( m = q, n = p \) is necessary for the existence of solutions when \( mn = pq \). Assume \( m > q \) and thus \( n < p \), and let \((u,v)\) be a solution. By means of the iterative procedure Lemma 3.2, we obtain (see (3.2)) \( u \leq -\frac{p}{m} \log a_n + L_{m, \beta_{n-1}} - a_n \log d(x), \quad v \geq \log a_{n+1} - \beta_n \log d(x) \), where \( \beta_n = \frac{2(m-q)}{mn} + \frac{pq}{mn} \beta_{n-1} \).

Hence, \( \beta_n \rightarrow +\infty \) and \( \alpha_n \rightarrow -\infty \). Let \( n \) be the minimum positive integer such that \( q \beta_n \geq 2 \). We deduce \( \Delta u \geq a_{n+1}^p d^{-\beta_n} e^{mu} \) in \( \Omega \), and it follows from Theorem 1.2 in [2] that \( u \) is bounded. Thus, no solution can exist. In the same way we rule out the possibility \( m < q \), and thus \( m = q, n = p \).
To show that the existence in this case is simpler, we look for a solution with $u = v$, and we find that $u$ has to satisfy $\Delta u = e^{(m+n)u}$ in $\Omega$ and $u = +\infty$ on $\partial\Omega$. That is $u = v = u_{m+n,0}$. It is easy to show that $(u_{m+n,0} + \log \epsilon, u_{m+n,0} - \frac{m}{n} \log \epsilon)$ is a solution to (1.1) for $\epsilon > 0$. □

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References


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