Admissible Perturbations of Differential Expressions
with Exponentially Decaying Coefficients
Preserving the Nullities

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Abstract

The essential spectrum and nullities of differential expressions of the form:
\[ \nu_0 = \sum_{k=0}^{r} b_k e^{\beta k s} D_{gk} \]

have been classified. A class of relatively compact perturbations of the above expressions which do not alter these properties has also been defined in [2]. Recently, admissible perturbations of this expression which preserve the essential spectrum have been classified [3]. In this paper, we define a class of admissible perturbations for the above form of differential expressions which preserves the nullities of these expressions.

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1 Special Expressions

A special expression is a differential expression in $L_2[1, \infty)$ of the form

$$M_0 = \sum_{k=0}^{r} b_k t^{\alpha_k} D_t^{\rho_k}$$

(1)

where $D_t = \frac{d}{dt}$, $\rho_k \in \mathbb{N}_0$ with $0 = \rho_0 < \rho_1 < \cdots < \rho_r = n$ and $\alpha_k \in \mathbb{R}$ for every $k$, satisfying $\alpha_0 = 0$, $\alpha_1 \leq \rho_1$, and

$$1 \geq \frac{\alpha_k - \alpha_{k-1}}{\rho_k - \rho_{k-1}} \geq \frac{\alpha_{k+1} - \alpha_k}{\rho_{k+1} - \rho_k},$$

(2)

for $k = 1, \ldots, r - 1$ if $r > 1$. We denote by $\sigma_1 < \sigma_2 < \cdots < \sigma_{s-1}$ those indices $k$ ($k = 1, 2, \ldots, r - 1$) for which the strong inequality holds in (2) and with $\sigma_0 = 0$ and $\sigma_s = r$. The coefficients $b_k$ are nonzero real numbers for $k = \sigma_1, \ldots, r$ satisfying

$$c_\sigma = \sum_{\rho_{\sigma_i} \leq \lambda \leq \rho_{\sigma_i+1}} (-1)^{\rho_\lambda+s} b_\lambda \geq 0$$

(3)

where $\rho_{\sigma_i} \leq \sigma \leq \rho_{\sigma_{i+1}}$, $i = 1, \ldots, s - 1$. For $k = 0, \ldots, \sigma_1 - 1$, $b_k$ may be an arbitrary complex constant. The indices $\sigma_1, \ldots, \sigma_s$ are called the kink indices, and the essential part of $M_0$ is $M_{0,0} = \sum_{k=0}^{\sigma_1} b_k t^{\alpha_k} D_t^{\rho_k}$.

If we plot the points $(\rho_k, \alpha_k)$, $k = 0, 1, \ldots, r$ on the Cartesian plane and connect $(\rho_0, \alpha_k)$ and $(\rho_{k+1}, \alpha_{k+1})$, we obtain the polygonal path generated by $M_0$. This path corresponds to the graph of the function $\gamma : [0, n] \rightarrow \mathbb{R}$ which we define, for $k \in [\rho_{\sigma_i}, \rho_{\sigma_{i+1}}]$, as

$$\gamma(k) = \frac{1}{\rho_{\sigma_{i+1}} - \rho_{\sigma_i}} \left\{ (k - \rho_{\sigma_i}) \alpha_{\sigma_{i+1}} + (\rho_{\sigma_{i+1}} - k) \alpha_{\sigma_i} \right\}.$$  

(4)

A class of relatively compact perturbations of special expressions are expressions of the form

$$M_1 = \sum_{j=0}^{n} r_j(t) D_t^j$$

(5)

where $r_j(t) \in C^j([1, \infty))$, $j = 0, \ldots, n$. and for $j = 0, \ldots, n$,

$$r_j(t) = o(t^{\gamma(j)})$$

(6)

as $t \rightarrow \infty$. For the invariance of nullities, we also admit perturbations of the above form satisfying

$$r_j^{(m)}(t) = o(t^{\gamma(j-m)})$$

(7)
as $t \to \infty$, for $j = 0, \ldots, n$ and $m = 0, \ldots, j$.

In [4], Schultze evaluated the essential spectrum and nullities of special expressions together with their relatively compact perturbations. These results are summarized in the following theorem.

**Theorem 1.1** Let $M_0$ be a special expression and $M_1$ a relatively compact perturbation of $M_0$ of the form (5) satisfying (6). Then

$$\sigma_e(M_0 + M_1) = \sigma_e(M_0) = \sigma_e(M_{0,0})$$

where

$$\sigma_e(M_{0,0}) = \begin{cases} \{ \sum_{k=0}^{\sigma_1} b_k z^{\rho_k} \mid \Re z = 0 \}, & \text{if } \alpha_1 < \rho_1 \\ \{ \sum_{k=0}^{\sigma_1} b_k \prod_{j=0}^{k-1} (z - (j + \frac{1}{2})) \mid \Re z = 0 \}, & \text{if } \alpha_1 = \rho_1. \end{cases}$$

If $M_1$ satisfies (7), then for every $\lambda \in \mathbb{C}\setminus \sigma_e(M_0)$,

$$\text{nul}(M_0 + M_1 - \lambda) = \text{nul}(M_0 - \lambda) = \text{nul}(M_{0,0} - \lambda) + \sum_{i=1}^{s-1} \# \{ z \mid \sum_{k=\sigma_{i+1}}^{\sigma_i} b_k z^{\rho_k} = 0, \Re z < 0 \}$$

where

$$\text{nul}(M_{0,0} - \lambda) = \begin{cases} \# \{ z \mid \sum_{k=0}^{\sigma_1} b_k z^{\rho_k} = \lambda, \Re z < 0 \}, & \text{if } \alpha_1 < \rho_1 \\ \# \{ z \mid \sum_{k=0}^{\sigma_1} b_k \prod_{j=0}^{k-1} (z - (j + \frac{1}{2})) = \lambda, \Re z < 0 \}, & \text{if } \alpha_1 = \rho_1. \end{cases}$$

Mumpar-Victoria [1] was able to describe a new type of perturbations, called the admissible perturbations, which do not necessarily satisfy (6) but preserve the essential spectrum and nullities.

**Definition 1.2** Let $M$ be a differential expression of the form

$$M = \sum_{j=0}^{n-1} r_j(t) D_t^j. \quad (8)$$

We say that $M$ is an admissible perturbation of the special expression $M_0$ if there exists a $B$ such that the coefficients $r_j(t)$ satisfy the following

$$\sup_{[x, x+1] \subset [1, \infty)} \int_x^{x+1} \left| \frac{r_j(t)}{w_j(t)} \right|^2 dt < B \quad (9)$$
where \( r_j(t) \in C^i([1, \infty)) \) for \( j = 0, \ldots, n-1 \) and \( 0 < w_j(t) \in C^\infty([1, \infty)) \) is an auxiliary function satisfying \( w_j(t) = o(t^{\gamma(j+1)}) \) and \( w_j(t) = o(t^{\gamma(j)}) \) as \( t \to \infty \).

For the invariance of nullities, we can only admit a somewhat less general class of perturbations consisting of expressions (8) satisfying

\[
\sup_{[x,x+1] \subset [1, \infty)} \int_x^{x+1} \left| \frac{r_{j-b}(t)}{w_b(t)} \right|^2 dt < \tilde{B}
\]

for \( b = 0, \ldots, n-1 \) and \( j = b, \ldots, n-1 \).

**Theorem 1.3** Let \( M_0 \) be a special expression and \( M \) an admissible perturbation of \( M_0 \) of the form (8) satisfying (9). Then \( \sigma_e(M_0 + M) = \sigma_e(M_0) \).

In addition, if \( M \) satisfies (10), then, for every \( \lambda \in \mathbb{C} \setminus \sigma_e(M_0) \),

\[
nul(M_0 + M - \lambda) = nul(M_0 - \lambda)
\]

2 Differential Expressions with Exponentially Decaying Coefficients

Let us now consider differential expressions of the form

\[
\nu_0 = \sum_{k=0}^{r} b_k e^{\beta_k s} D^\rho_k
\]

where \( D_s = \frac{d}{ds} \), \( \rho_k \in \mathbb{N}_0 \) for every \( k \), such that \( 0 = \rho_0 < \rho_1 < \cdots < \rho_r = n \), and \( \beta_k \in \mathbb{R} \), satisfying

\[
\beta_0 = 0 \quad \text{and} \quad 0 \geq \frac{\beta_k - \beta_{k-1}}{\rho_k - \rho_{k-1}} \geq \frac{\beta_{k+1} - \beta_k}{\rho_{k+1} - \rho_k},
\]

for \( k = 1, \ldots, r-1 \) if \( r > 1 \), and the coefficients \( b_k \) satisfy (3). We denote by \( \sigma_1 < \sigma_2 < \cdots < \sigma_{s-1} \) those indices \( k \), for which strict inequality holds in (12) with \( \sigma_0 = 0 \) and \( \sigma_s = r \). The polygonal path generated by \( \nu_0 \) with the aforementioned conditions lies on or below the horizontal axis and it is given by the graph of the function \( \tilde{\gamma} : [0, n] \to \mathbb{R} \) defined, for \( k \in [\rho_{\sigma_i}, \rho_{\sigma_i+1}] \), as

\[
\tilde{\gamma}(k) = \frac{1}{\rho_{\sigma_{i+1}} - \rho_{\sigma_i}} \{(k - \rho_{\sigma_i})\beta_{\sigma_{i+1}} + (\rho_{\sigma_{i+1}} - k)\beta_{\sigma_i}\}.
\]

**Remark 2.1** For every \( k \), \( \tilde{\gamma}(\rho_k) = \beta_k \) and if \( k > j \), then \( \tilde{\gamma}(k) \leq \tilde{\gamma}(j) \). If \( \beta_{k+1} < \beta_k \) for \( k = 0, \ldots, r-1 \), then for \( k > j \), \( \tilde{\gamma}(k) < \tilde{\gamma}(j) \).

Let \( \nu_1 \) be a differential expression of the form

\[
\nu_1 = \sum_{k=0}^{n} r_k(s) D^k_s
\]
Admissible perturbations of differential expressions preserving the nullities

where \( r_k(s) \in C^k([0, \infty)), k = 0, \ldots, n \). We say that \( \nu_1 \) is a \textit{relatively compact perturbation} of the differential expression \( \nu_0 \) if for every \( k = 0, \ldots, n \),

\[
r_k(s) = o(e^{\tilde{\gamma}(k)s})
\]

as \( s \to \infty \). For the invariance of the nullities, we admit a class of perturbations consisting of expressions \( \nu_1 \) of the form (13) satisfying

\[
r_k^{(m)}(s) = o(e^{\tilde{\gamma}(k-m)s})
\]

as \( s \to \infty \), for \( k = 0, \ldots, n \) and \( m = 0, \ldots, k \).

The following theorem, due to Roque [2], gives the classification of the spectral properties of \( \nu_0 \) together with its relatively compact perturbation.

**Theorem 2.2** Let \( \nu_0 \) be a differential expression of the form (11) satisfying (12), and let the coefficients \( b_k \) satisfy (3). Let \( \nu_1 \) be a relatively compact perturbation of \( \nu_0 \) of the form (13) satisfying (14). Then

\[
\sigma_e(\nu_0 + \nu_1) = \sigma_e(\nu_0) = \{ \sum_{k=0}^{\sigma_1} b_k z^{\rho_k} \mid Re \, z = 0 \}.
\]

Furthermore, if \( \nu_1 \) satisfies (15), then for every \( \lambda \in \mathbb{C} \setminus \sigma_e(\nu_0) \),

\[
nul(\nu_0 + \nu_1 - \lambda) = nul(\nu_0 - \lambda)
\]

\[
= \# \{ z \mid \sum_{k=0}^{\sigma_1} b_k z^{\rho_k} = \lambda, Re \, z < 0 \} + \sum_{i=1}^{s-1} \# \{ z \mid \sum_{k=\sigma_i}^{\sigma_{i+1}} b_k z^{\rho_k} = 0, Re \, z < 0 \}.
\]

In the proof of Theorem 2.2, \( \nu_0 \) was transformed to a special expression using the transformation \( \eta : L_2[1, \infty) \to L_2[0, \infty) \) defined as \( (\eta f)(s) = e^{s/2} f(e^s) \) where \( 0 \leq s < \infty \). The mapping \( \eta \) is a surjective linear isometry and its inverse map \( \eta^{-1} \) is given by \( (\eta^{-1} g)(t) = t^{-1/2} g(\log t) \), where \( 1 \leq t < \infty \). We now describe the transformation of a differential expression under the mapping \( \eta \).

**Lemma 2.3** Let \( \tau = \sum_{k=0}^{r} h_k(t) D_t^{\rho_k}, \) where \( h_k(t) \in C^k([1, \infty)) \) and \( \rho_k \in \mathbb{N}_0 \) such that \( 0 = \rho_0 < \rho_1 < \cdots < \rho_r = n \). Then

\[
\eta \tau \eta^{-1} = h_0(e^s) + \sum_{k=1}^{r} h_k(e^s) e^{-\rho_k s} \prod_{j=0}^{\rho_k-1} (D_s - (j + \frac{1}{2})).
\]
3 Main Results

The aim of this paper is to determine a new class of perturbations for \( \nu_0 \) which preserves its nullities. We now define the admissible perturbations of \( \nu_0 \).

**Definition 3.1** Let \( \nu_0 \) be a differential expression of the form (11) satisfying (12). An admissible perturbation of \( \nu_0 \) is an expression \( \nu \) in \( \mathcal{L}_2([0, \infty)) \) of the form

\[
\nu = \sum_{j=0}^{n-1} h_j(s) D_j^s
\]

where \( h_j(s) \in C^j([0, \infty)) \) for every \( j \) satisfying:

**AP1.** For every \( i > j \) and for some finite \( C \geq 0 \), \( \left| \frac{h_i(s)}{h_j(s)} \right| \leq C \).

**AP2.** There exists a \( D > 0 \) such that

\[
sup_{x \leq x+1 \in [1, \infty)} \int_{\log x}^{\log(x+1)} e^s \left| \frac{h_i(s)}{q_j(s)} \right|^2 ds < D
\]

where \( 0 < q_j(s) \in C^\infty([0, \infty)) \) for all \( j \) and \( q_j(s) = o(e^{5(j+1)s}) \) as \( s \to \infty \).

The functions \( q_j(s) \) are called auxiliary functions.

**Theorem 3.2** Let \( \nu_0 \) be a differential expression of the form (11) satisfying (12), and let the coefficients \( b_k \) satisfy (3). If \( \nu \), of the form (16), is an admissible perturbation of \( \nu_0 \), then \( \sigma_e(\nu_0 + \nu) = \sigma_e(\nu_0) \).

Conditions AP1 and AP2 are sufficient to preserve the essential spectrum, as shown in [3], but not the nullity of \( \nu_0 \). We will now define conditions AP3, AP4, and AP5 for the admissible perturbation of the expression \( \nu_0 \) and prove that these conditions are sufficient to preserve the nullity of this expression.

We will consider expressions \( \nu \) of the form (16) satisfying:

**AP3.** For every \( i > j \), \( 0 \leq k \leq j - b \) and for some finite \( \tilde{C} \geq 0 \), \( \left| \frac{h_{i+k}(s)}{h_j(s)} \right| \leq \tilde{C} \).

**AP4.** \( h_j^{(j-b)}(s) = sup_{0 \leq k \leq j-b} \{ h_j^{(k)}(s) \} \).

**AP5.** There exists a \( \tilde{D} > 0 \) such that

\[
sup_{x \leq x+1 \in [1, \infty)} \int_{\log x}^{\log(x+1)} e^s \left| \frac{h_j^{(j-b)}(s)}{q_b(s)} \right|^2 ds < \tilde{D}
\]

where \( b = 0, 1, \ldots, n - 1 \) and \( j = b, \ldots, n - 1 \) and \( q_b(s) \) are the auxiliary functions for the admissible perturbation \( \nu \) in Definition 3.1.
In the following lemma, we show that transforming $\nu$ under $\eta$ results into an admissible perturbation of a special expression.

**Lemma 3.3** Let $\nu_0$ be a differential expression of the form (11) satisfying (12), and let the coefficients $b_k$ satisfy (3). If $\nu$ is an admissible perturbation of $\nu_0$ satisfying $AP3$, $AP4$, and $AP5$, then $\tau = \eta^{-1} \nu \eta$ is an admissible perturbation of $\tau_0 = \sum_{k=0}^{r} b_k t_k^{m_k} D_t^{m_k}$ satisfying (10) where $\alpha_k = \beta_k + \rho_k$.

**Proof:** We show that $\tau = \eta^{-1} \nu \eta = \sum_{j=0}^{n-1} r_j(t) D_t^j$ where $r_j(t) = \sum_{i=j}^{n-1} c_{ji} h_i(\log t) t^j$ satisfies (10). Note that,

$$r_j^{(m)}(t) = \sum_{i=j}^{n-1} c_{ji} [h_i(\log t) t^j]^{(m)} = \sum_{i=j}^{n-1} c_{ji} \sum_{k=0}^{m} J_{j,k}^{m} h_i^{(k)}(\log t) t^{j-m}$$

where $c_{ji}$ and $J_{j,k}^{m}$ are constants. Since $r_j^{(m)}(t) = \sum_{i=j}^{n-1} c_{ji} \sum_{k=0}^{m} J_{j,k}^{m} h_i^{(k)}(\log t) t^{j-m}$, and taking the auxiliary functions $w_b(t) = q_b(\log t) t^b$, we have

$$\int_{x}^{x+1} \left| \frac{r_j^{(j-b)}(t)}{w_b(t)} \right|^2 dt = \int_{\log x}^{\log (x+1)} e^s \left| \sum_{i=j}^{n-1} c_{ji} \sum_{k=0}^{j-b} J_{j,k}^{m} h_i^{(k)}(s) \right|^2 ds, \quad \log t = s$$

$$\leq J^2 c^2 \int_{\log x}^{\log (x+1)} e^s \left\{ \sum_{i=j}^{n-1} \left| h_i^{(k)}(s) \right| q_b(s) \right\}^2 ds$$

where $J = \sup_{0 \leq k \leq j-b} \{ J_{j,k}^{j-b} \}$ and $c = \sup_{i \geq j} \{ c_{ji} \}$. From $AP3$, $\left| h_i^{(k)}(s) \right| \leq \bar{C} \left| h_j^{(k)}(s) \right|$, for $i > j$, and from $AP4$ $h_j^{(j-b)}(s) = \sup_{0 \leq k \leq j-b} \{ h_j^{(k)}(s) \}$. Thus,

$$\int_{x}^{x+1} \left| \frac{r_j^{(j-b)}(t)}{w_b(t)} \right|^2 dt < J^2 c^2 \int_{\log x}^{\log (x+1)} e^s \left\{ \bar{C} \sum_{i=j}^{n-1} \left| h_j^{(k)}(s) \right| q_b(s) \right\}^2 ds$$

$$= J^2 c^2 \bar{C} \int_{\log x}^{\log (x+1)} e^s \left\{ \sum_{i=j}^{n-1} \sum_{k=0}^{j-b} \left| h_j^{(k)}(s) \right| q_b(s) \right\}^2 ds$$

$$< J^2 c^2 \bar{C} \int_{\log x}^{\log (x+1)} e^s \left\{ \sum_{i=j}^{n-1} \sum_{k=0}^{j-b} \left| h_j^{(j-b)}(s) \right| q_b(s) \right\}^2 ds$$

$$= J^2 c^2 \bar{C} \int_{\log x}^{\log (x+1)} e^s \left\{ \left| h_j^{(j-b)}(s) \right| q_b(s) \right\}^2 ds$$

$$= J^2 c^2 \bar{C} (n-j)^2 (j-b+1) \int_{\log x}^{\log (x+1)} e^s \left| h_j^{(j-b)}(s) \right|^2 ds.$$
Letting $A = J^2c^2\tilde{C}^2(n - j)^2(j - b + 1)^2$, then taking the supremum of both sides on the interval $[x, x + 1] \subset [1, \infty)$, we get

$$\sup_{[x, x+1] \subset [1, \infty)} \int_x^{x+1} \left| \frac{r_j^{(j-b)}(t)}{w_b(t)} \right|^2 dt < A \sup_{[x, x+1] \subset [1, \infty)} \int_{\log x}^{\log(x+1)} e^s \left| \frac{h_j^{(j-b)}(s)}{q_b(s)} \right|^2 ds.$$  

From AP5, for $b = 0, 1, ..., n - 1$ and $j = b, ..., n - 1$,

$$\sup_{[x, x+1] \subset [1, \infty)} \int_{\log x}^{\log(x+1)} e^s \left| \frac{h_j^{(j-b)}(s)}{q_b(s)} \right|^2 ds < \tilde{D}.$$  

Taking $A\tilde{D} = \tilde{B}$, we have

$$\sup_{[x, x+1] \subset [1, \infty)} \int_x^{x+1} \left| \frac{r_j^{(j-b)}(t)}{w_b(t)} \right|^2 dt < \tilde{B}.$$  

This proves the assertion.

From the above lemma, we have the main result

**Theorem 3.4** Let $\nu_0$ be a differential expression of the form (11) satisfying (12), and let the coefficients $b_k$ satisfy (3). If $\nu$ is an admissible perturbation of $\nu_0$ satisfying AP3, AP4, and AP5, then, for any $\lambda \in \mathbb{C} \setminus \sigma_e(\nu_0)$,

$$\text{nul}(\nu_0 + \nu - \lambda) = \text{nul}(\nu_0 - \lambda).$$

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