Some Generalizations on Generalized Topology and Minimal Structure Spaces

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Abstract

The aim of this paper is to introduce some generalizations for closed sets in generalized topology and minimal structure spaces. We investigate some properties of these sets on these spaces. Moreover, we give the concept of \( T_0 \)-GTMS spaces, \( T_{1/2} \)-GTMS spaces and \( R_0 \)-GTMS spaces, and the various relationships between them.

Keywords: Generalized topology and minimal structure space; \( G \)-closed; \( G^* \)-closed; \( T_0 \)-GTMS space, \( T_{1/2} \)-GTMS space and \( R_0 \)-GTMS space

1 introduction

The concept of minimal structure (briefly \( m \)-structure) was introduced by V. Popa and T. Noiri [9] in 2000. Also they introduced the notions of \( m_X \)-open sets and \( m_X \)-closed sets and characterize those sets using \( m_X \)-closure and \( m_X \)-operators, respectively. T. Noiri and V. Popa [8] obtained the definitions and characterizations of separation axioms by using the concept of minimal structure. Á. Csařszař [2] introduced the concept of generalized neighborhood systems and generalized topological spaces. He also introduced the concepts of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. In particular, he investigated characterizations for the generalized continuous function by using a closure operator defined on generalized neighborhood systems. Moreover he studied the simplest separation axioms for generalized topologies in [3].
Recently, in 2011, S. Buadong and et al. [1] introduced the notion of the generalized topology and minimal structure spaces (briefly GTMS). They studied some properties of closed sets on the space. Also, they introduced the concepts of the separation axioms $T_1$ and $T_2$ on generalized topology and minimal structure spaces.

We present this paper to shed light on other types of sets on the generalized topology and minimal structure spaces which followed the generalized closed sets defined by Levin [5], where they gained the attention of many researchers since then, and these types included bitopological spaces. By using these sets, the separation axioms between the axioms $T_1$ and $T_0$ was defined, which is a $T_{1/2}$-axiom.

Through this paper, we introduce the concepts of $gmG$-closed sets, $mgG$-closed sets, $G$-closed sets, $gG$-closed sets, $mG$-closed sets and $G^*$-closed sets for two reasons. The first reason is that the bitopological space is a special setting of the generalized topology and minimal structure spaces. In a recent study, K. kannan [4] highlighted the importance applications for some types of generalized closed sets on bitopological spaces. He studied the nature and properties of some generalized closed sets in the bitopological spaces associated to the digraph. The second reason is that to define new separation axioms on the generalized topology and minimal structure spaces, for example $T_0$ and $T_{1/2}$ axioms. We define these axioms in an appropriate manner so located $T_{1/2}$ between the axioms $T_1$ and $T_0$.

In this paper, we faced some challenges because we had to fined a new characterization for $T_1$-axiom by $T_0$ and $R_0$ axioms. But we overcame this difficulty by introducing the definition of $G^*$-closed set and then we used it to define the $R_0$-axiom.

## 2 Preliminaries

**Definition 2.1.** [2] Let $X$ be a nonempty set and $g$ a collection of subsets of $X$. Then $g$ is called a generalized topology (briefly $GT$) on $X$ if and only if $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $\bigcup_{i \in I} G_i \in g$. We call the pair $(X, g)$ a generalized topological space (briefly $GTS$) on $X$. The elements of $g$ are called $g$-open sets and the complements are called $g$-closed sets.

The closure of a subset $A$ in a generalized topological space $(X, g)$, denoted by $c_g(A)$, is the intersection of generalized closed sets including $A$, i.e., the smallest $g$-closed set containing $A$. The interior of $A$, denoted by $i_g(A)$, is the union of generalized open sets contained in $A$, i.e., the largest $g$-open set contained in $A$.

**Proposition 2.1.** [7] Let $(X, g)$ be a generalized topological space. For subsets $A$ and $B$ of $X$, the following properties hold.
Lemma 2.2. \[6\] Let \((X, g)\) be a generalized topological space and \(A \subseteq X\).
Then
(1) \(x \in i_g (A)\) if and only if there exists \(V \in g\) such that \(x \in V \subseteq A\);
(2) \(x \in c_g (A)\) if and only if \(V \cap A \neq \emptyset\) for every \(g\)-open set \(V\) containing \(x\).

Definition 2.3. \[9\] Let \(X\) be a nonempty set and \(P (X)\) the power set of \(X\). A subfamily \(m\) of \(P (X)\) is called a minimal structure (briefly \(m\)-structure) on \(X\) if \(\emptyset \in m\) and \(X \in m\).

By \((X, m)\), we denote a nonempty set \(X\) with an \(m\)-structure \(m\) on \(X\) and it is called an \(m\)-space. Each member of \(m\) is said to be \(m\)-open and the complement of an \(m\)-open set is said to be \(m\)-closed.

Lemma 2.1. \[6\] Let \(X\) be a nonempty set and \(m\) a minimal structure on \(X\). For a subset \(A\) of \(X\), the \(m\)-closure of \(A\) denoted by \(c_m (A)\) and the \(m\)-interior of \(A\) denoted by \(i_m (A)\), are defined as follows:
(1) \(c_m (A) = \cap \{F : A \subseteq F, X \setminus F \in m\}\);
(2) \(i_m (A) = \cup \{U : U \subseteq A, U \in m\}\).

Lemma 2.2. \[6\] Let \(X\) be a nonempty set with a minimal structure \(m\) and \(A\) a subset of \(X\). Then \(x \in c_m (A)\) if and only if \(U \cap A \neq \emptyset\) for every \(m\)-open set \(U\) containing \(x\).

Definition 2.4. \[1\] Let \(X\) be a nonempty set and let \(g\) be a generalized topology and \(m\) a minimal structure on \(X\). A triple \((X, g, m)\) is called a generalized topology and minimal structure space (briefly GTMS space).

Definition 2.5. \[1\] Let \((X, g, m)\) be a GTMS space. A subset \(A\) of \(X\) is said to be a \(gm\)-closed if \(c_g (c_m (A)) = A\). A subset \(A\) of \(X\) is said to be a \(mg\)-closed if \(c_m (c_g (A)) = A\).
Lemma 2.3. [1] Let \((X, g, m)\) be a GTMS space and \(A \subseteq X\). Then
(1) \(A\) is \(gm\)-closed if and only if \(c_m(A) = A\) and \(c_g(A) = A\);
(2) \(A\) is \(mg\)-closed if and only if \(c_m(A) = A\) and \(c_g(A) = A\).

Proposition 2.3. [1] Let \((X, g, m)\) be a GTMS space and \(A \subseteq X\). Then \(A\) is \(gm\)-closed if and only if \(A\) is \(mg\)-closed.

Definition 2.6. [1] Let \((X, g, m)\) be a GTMS space and \(A\) a subset of \(X\). Then \(A\) is said to be \(\text{closed}\) if \(A\) is \(gm\)-closed. The complement of a closed set is said to be an \(\text{open set}\).

Proposition 2.4. [1] Let \((X, g, m)\) be a GTMS space. If \(A\) and \(B\) are closed, then \(A \cap B\) is closed.

Definition 2.7. [1] Let \((X, g, m)\) be a GTMS space. A subset \(A\) of \(X\) is said to be a \(s\)-\text{closed}\) if \(c_g(A) = c_m(A)\). A subset \(A\) of \(X\) is said to be a \(c\)-\text{closed}\) if \(c_g(c_m(A)) = c_m(c_g(A))\). The complement of a \(s\)-\text{closed} (resp. \(c\)-\text{closed}) set is called a \(s\)-\text{open} (resp. \(c\)-\text{open}) set.

Theorem 2.1. [1] Let \((X, g, m)\) be a GTMS space and \(A \subseteq X\). Then
(1) \(A\) is closed if and only if there exists a \(s\)-\text{closed} set \(B\) such that \(A = c_g(B)\);
(2) \(A\) is closed if and only if there exists a \(c\)-\text{closed} set \(B\) such that \(A = c_g(c_m(B))\).

Definition 2.8. [1] A GTMS space \((X, g, m)\) is called a \(T_1\)-GTMS space if for any pair distinct points \(x\) and \(y\) in \(X\), there exist a \(g\)-open set \(U\) and a \(m\)-open set \(V\) such that \(x \in U, y \notin U\) and \(y \in V, x \notin V\).

Theorem 2.2. [1] Let \((X, g, m)\) be a GTMS space. Then \(X\) is \(T_1\)-GTMS space if and only if every singleton subset of \(X\) is closed.

3 \(G\)-closed (\(G^*\)-closed) sets

Definition 3.1. Let \((X, g, m)\) be a GTMS space. A subset \(A\) of \(X\) is said to be a \(gmG\)-\text{closed} if \(c_g(c_m(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open. A subset \(A\) of \(X\) is said to be a \(mgG\)-\text{closed} if \(c_m(c_g(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open. A subset \(A\) of \(X\) is said to be a \(G\)-\text{closed} if \(A\) is \(mgG\)-\text{closed} and \(gmG\)-\text{closed}.

Theorem 3.1. Let \((X, g, m)\) be a GTMS space. Then every closed set is \(G\)-\text{closed}.

Proof. Let \(A\) be a closed subset of \(X\) such that \(A \subseteq U\) and \(U\) is open. Since \(A\) is closed, then \(A\) is \(gm\)-closed and hence \(c_g(c_m(A)) = A \subseteq U\). Thus \(A\) is \(gmG\)-closed. Also, \(A\) is \(mg\)-closed and hence \(c_m(c_g(A)) = A \subseteq U\). Thus \(A\) is \(mgG\)-closed. Therefore \(A\) is \(G\)-closed. \(\square\)
Remark. The converse of Theorem 3.1 is not true by the following example.

Example 3.1. Let $X = \{a, b, c, d\}$ with generalized topology $g = \{\emptyset, \{a, b\}, \{a, d\}, \{a, b, d\}\}$ and minimal structure $m = \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, d\}, X\}$. Then $A = \{a, b, c\}$ is $G$-closed but not closed.

Remark. The finite intersection of $G$-closed sets need not be $G$-closed as shown by the following example.

Example 3.2. Let $X = \{a, b, c, d\}$ with generalized topology $g = \{\emptyset, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$ and minimal structure $m = \{\emptyset, \{a, b\}, \{a, b, c\}, \{a, d\}, X\}$. Then $\{a, c\}$ and $\{a, d\}$ are $G$-closed but $\{a, c\} \cap \{a, d\} = \{a\}$ is not $G$-closed.

Definition 3.2. Let $(X, g, m)$ be a GTMS space. A subset $A$ of $X$ is said to be a $gG$-closed if $c_m(c_g(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open. A subset $A$ of $X$ is said to be a $mgG$-closed if $c_g(c_m(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is $m$-open. A subset $A$ of $X$ is said to be a $G^*$-closed if $A$ is $gG$-closed and $mgG$-closed.

Theorem 3.2. Let $(X, g, m)$ be a GTMS space. Then every closed set is $G^*$-closed.

Proof. Similar to the proof of Theorem 3.1.

Remark. The converse of the above theorem need not be true as seen from the following example.

Example 3.3. Let $X = \{a, b, c\}$ with generalized topology $g = \{\emptyset, \{b\}, \{a, b\}\}$ and minimal structure $m = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then $\{b, c\}$ is $G^*$-closed but not closed.

Theorem 3.3. Let $(X, g, m)$ be a GTMS space and $A \subseteq X$. Then the following hold.

(1) If $A$ is $gmgG$-closed and $F$ is closed, then $A \cap F$ is $gmgG$-closed.
(2) If $A$ is $mgG$-closed and $F$ is closed, then $A \cap F$ is $mgG$-closed.
(3) If $A$ is $gG$-closed and $F$ is closed, then $A \cap F$ is $gG$-closed.

Proof. (1) Let $U$ be open set such that $A \cap F \subseteq U$. Then $A \subseteq U \cup (X \setminus F)$. Since $A$ is $gmgG$-closed, then $c_g(c_m(A)) \subseteq U \cup (X \setminus F)$. Thus $c_g(c_m(A)) \cap F \subseteq U$. Hence $A \cap F$ is $gmgG$-closed.
(2), (3) Similar to the proof of the part (1).

Corollary 3.4. Let $(X, g, m)$ be a GTMS space and $A \subseteq X$. If $A$ is $G$-closed and $F$ is closed, then $A \cap F$ is $G$-closed.
Theorem 3.5. Let \((X, g, m)\) be a GTMS space and \(A \subseteq X\). Then the following hold.

1. If \(A\) is open and \(gmG\)-closed, then \(A\) is closed.
2. If \(A\) is open and \(mgG\)-closed, then \(A\) is closed.
3. If \(A\) is \(g\)-open and \(gG\)-closed, then \(A\) is closed.
4. If \(A\) is \(m\)-open and \(mG\)-closed, then \(A\) is closed.

Proof. We only prove (1) and the proofs (2), (3), (4) follow the same way.

(1) Suppose a subset \(A\) of \(X\) is both open and \(gmG\)-closed. Then \(c_g(c_m(A)) \subseteq A\). Since \(A \subseteq c_g(c_m(A))\), we have \(c_g(c_m(A)) = A\). Thus \(A\) is closed in \(X\).

Corollary 3.6. Let \((X, g, m)\) be a GTMS space and \(A \subseteq X\). Then the following hold.

1. If \(A\) is open and \(G\)-closed, then \(A\) is closed.
2. If \(A\) is open and \(G^*\)-closed, then \(A\) is closed.

Theorem 3.7. Let \((X, g, m)\) be a GTMS space and \(A \subseteq X\). Then the following hold.

1. If \(A\) is \(gmG\)-closed, then \(c_g(c_m(A)) \setminus A\) does not contain any nonempty closed set.
2. If \(A\) is \(mgG\)-closed, then \(c_m(c_g(A)) \setminus A\) does not contain any nonempty closed set.
3. If \(A\) is \(gG\)-closed, then \(c_m(c_g(A)) \setminus A\) does not contain any nonempty \(g\)-closed set.
4. If \(A\) is \(mG\)-closed, then \(c_g(c_m(A)) \setminus A\) does not contain any nonempty \(m\)-closed set.

Proof. We only prove (1) and the proofs (2), (3), (4) follow the same way.

(1) Assume that \(F\) is a nonempty closed subset of \(c_g(c_m(A)) \setminus A\). Clearly \(A \subseteq X \setminus F\), where \(A\) is \(gmG\)-closed and \(X \setminus F\) is open. Thus \(c_g(c_m(A)) \subseteq X \setminus F\), that is \(F \subseteq (X \setminus c_g(c_m(A)))\). Since \(F \subseteq c_g(c_m(A))\), then \(F \subseteq (X \setminus c_g(c_m(A))) \cap c_g(c_m(A)) = \emptyset\).

Corollary 3.8. Let \((X, g, m)\) be a GTMS space and \(A \subseteq X\). Then the following hold.

1. If \(A\) is \(G\)-closed, then \(c_g(c_m(A)) \setminus A\) and \(c_m(c_g(A)) \setminus A\) do not contain any nonempty closed set.
2. If \(A\) is \(G^*\)-closed, then \(c_m(c_g(A)) \setminus A\) does not contain any nonempty \(g\)-closed set and \(c_g(c_m(A)) \setminus A\) does not contain any nonempty \(m\)-closed set.

Theorem 3.9. Let \((X, g, m)\) be a GTMS space and \(A, B \subseteq X\). Then the following hold.

1. If \(A\) is \(gmG\)-closed and \(A \subseteq B \subseteq c_m(A)\), then \(B\) is \(gmG\)-closed.
(2) If $A$ is $mgG$-closed and $A \subseteq B \subseteq c_g(A)$, then $B$ is $mgG$-closed.
(3) If $A$ is $gG$-closed and $A \subseteq B \subseteq c_g(A)$, then $B$ is $gG$-closed.
(4) If $A$ is $mgG$-closed and $A \subseteq B \subseteq c_m(A)$, then $B$ is $mgG$-closed.
(5) If $A$ is $gmG$-closed and $c_g(B) \subseteq A \subseteq B$, then $B$ is $gmG$-closed.
(6) If $A$ is $mgG$-closed and $c_m(B) \subseteq A \subseteq B$, then $B$ is $gmG$-closed.

Proof. (1) Let $B \subseteq U$, where $U$ is open. Since $A \subseteq B$, then $A \subseteq U$. Thus $c_g(c_m(A)) \subseteq U$, since $A$ is $gmG$-closed. Moreover, $c_m(B) \subseteq c_m(c_m(A)) = c_m(A)$, since $B \subseteq c_m(A)$. Thus $c_g(c_m(B)) \subseteq c_g(c_m(A)) \subseteq U$. This shows that $B$ is $gmG$-closed.
(2), (3), (4) Similar to the proof of the part (1).
(5) Let $B \subseteq U$, where $U$ is open. Since $A \subseteq B$, then $A \subseteq U$. Thus $c_g(c_m(A)) \subseteq U$, since $A$ is $gmG$-closed. Moreover, $c_m(c_g(B)) \subseteq c_m(A)$, since $c_g(B) \subseteq A$. Thus $c_m(c_g(B)) \subseteq c_g(c_m(c_g(B))) \subseteq c_g(c_m(A)) \subseteq U$. This shows that $B$ is $mgG$-closed.
(6) Similar to the proof of the part (5). 

4 Lower separation axioms

Definition 4.1. A GTMS space $(X, g, m)$ is called a $T_{1/2}$-GTMS space if every $gmG$-closed set is closed.

Theorem 4.1. Let $(X, g, m)$ be a GTMS space. Then the following are equivalent.
(1) $X$ is a $T_{1/2}$-GTMS space.
(2) Every singleton of $X$ is either open or closed.

Proof. (1) $\implies$ (2) Let $X$ be a $T_{1/2}$-GTMS space and $x \in X$. If $x$ is not closed, then $X \setminus \{x\}$ is not open and then $X \setminus \{x\}$ is triviality $gmG$-closed. Thus $X \setminus \{x\}$ is closed and hence $\{x\}$ is open.
(2) $\implies$ (1) Let $A$ be a $gmG$-closed and $x \in c_g(c_m(A))$. We have the following two cases:
Case (i): $\{x\}$ is closed. By Theorem 3.7(1), $c_g(c_m(A)) \setminus A$ does not contain any nonempty closed set. This shows that $x \in A$.
Case (ii): $\{x\}$ is open. If $x \notin A$, then $A \subseteq X \setminus \{x\}$. Since $X \setminus \{x\}$ is closed, then $c_g(c_m(A)) \subseteq c_g(c_m(X \setminus \{x\})) = X \setminus \{x\}$. Thus $x \notin c_g(c_m(A))$.
In either case, $c_g(c_m(A)) = A$, that is $A$ is closed. Thus $X$ is a $T_{1/2}$-GTMS space.

Theorem 4.2. Let $(X, g, m)$ be a GTMS space. Then the following are equivalent.
(1) $X$ is a $T_{1/2}$-GTMS space.
(2) Every $mgG$-closed set is closed.
Proof. Similar to the proof of Theorem 4.1.

Corollary 4.3. Let \((X, g, m)\) be a GTMS space. Then the following hold.

(1) If \(X\) is a \(T_{1/2}\)-GTMS space, then every subset \(A\) of \(X\) is gmG-closed if and only if \(A\) is mgG-closed.

(2) \(X\) is a \(T_{1/2}\)-GTMS space if and only if every G-closed set is closed.

Theorem 4.4. Let \((X, g, m)\) be a \(T_{1/2}\)-space. Then the following hold.

(1) Every gG-closed set is closed.

(2) Every singleton of \(X\) is either open or g-closed.

Proof. (1) Suppose that \(A\) is gG-closed. Then \(A\) is mgG-closed. Since \(X\) is \(T_{1/2}\)-GTMS space, then \(A\) is closed, from Theorem 4.2.

(2) Let \(x \in X\). If \(x\) is not g-closed, then \(X \setminus \{x\}\) is not g-open and then \(X \setminus \{x\}\) is triviality gG-closed. By (1), \(X \setminus \{x\}\) is closed and hence \(\{x\}\) is open.

Remark. The converse of the above theorem need not be true, in general, as a simple we give the following example.

Example 4.1. Let \(X = \{a, b, c\}\) with generalized topology \(g = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}\) and minimal structure \(m = \{\emptyset, \{a\}, \{b\}, X\}\). It is easy to check that every gG-closed set is closed, and every singleton of \(X\) is either open or g-closed. But \(X\) is not \(T_{1/2}\)-GTMS space.

Theorem 4.5. A GTMS space \((X, g, m)\) is a \(T_{1/2}\)-GTMS space if and only if for every singleton \(\{x\}\) in \(X\), there exists either s-closed subset \(A\) such that \(\{x\} = c_g(A)\) or s-open subset \(B\) such that \(\{x\} = i_g(B)\).

Proof. Assume that \(X\) is a \(T_{1/2}\)-GTMS space. By Theorem 4.1, \(\{x\}\) is closed or open. If \(\{x\}\) is closed, By Theorem 2.1(1), there exists a s-closed subset \(A\) such that \(\{x\} = c_g(A)\). On the other hand, let \(\{x\}\) is open; that is \(X \setminus \{x\}\) is closed. Again by Theorem 2.1(1), there exists a s-closed subset \(C\) such that \(X \setminus \{x\} = c_g(C)\). Thus \(\{x\} = i_g(X \setminus C)\). Put \(B = X \setminus C\), We have a s-open subset \(B\) such that \(\{x\} = i_g(B)\).

Conversely, let \(x \in X\). We have the tow following cases:

Case (i): There exists a s-closed subset \(A\) such that \(\{x\} = c_g(A)\). By Theorem 2.1 (1), \(\{x\}\) is closed.

Case (ii): There exists a s-open subset \(B\) such that \(\{x\} = i_g(B)\). Thus \(X \setminus \{x\} = X \setminus (i_g(B)) = c_g(X \setminus B)\). Again by Theorem 2.1 (1), \(X \setminus \{x\}\) is closed, since \(X \setminus B\) is a s-closed. Thus \(\{x\}\) is open.

The two cases show that \(\{x\}\) is either closed or open. This shows that \(X\) is a \(T_{1/2}\)-GTMS space.

Theorem 4.6. A GTMS space \((X, g, m)\) is a \(T_{1/2}\)-GTMS space if and only if for every singleton \(\{x\}\) in \(X\), there exists either c-closed subset \(A\) such that \(\{x\} = c_g(c_m(A))\) or c-open subset \(B\) such that \(\{x\} = i_g(i_m(B))\).
Proof. Similar to the proof of Theorem 4.5. \qed

\textbf{Theorem 4.7.} Let \((X, g, m)\) be a \(T_{1/2}\)-GTMS space and \(x \in X\). If \(\{x\}\) is not \(g\)-closed, then the following hold.

(1) There exists a \(s\)-open subset \(A\) such that \(\{x\} = i_g(A)\).

(2) There exists a \(c\)-open subset \(A\) such that \(\{x\} = i_m(i_m(A))\).

Proof. The proof is immediate from Theorem 2.2 and Theorem 4.1. \qed

\textbf{Theorem 4.8.} Let \((X, g, m)\) be a GTMS space. If \(X\) is a \(T_1\)-GTMS space, then \(X\) is a \(T_{1/2}\)-GTMS space.

Proof. The proof is immediate from Theorem 2.2 and Theorem 4.1. \qed

\textbf{Remark.} The converse of Theorem 4.8 is not true. We can be seen from the following example.

\textbf{Example 4.2.} Let \(X = \{a, b, c\}\) with generalized topology \(g = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}\) and minimal structure \(m = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}\). Then \(X\) is a \(T_{1/2}\)-GTMS space but not \(T_1\)-GTMS space.

\textbf{Definition 4.2.} A GTMS space \((X, g, m)\) is called a \(T_0\)-space if for any pair of distinct points \(x\) and \(y\) in \(X\), there exists a subset \(U\) which is either \(g\)-open or \(m\)-open such that \(x \in U\), \(y \notin U\) or \(y \in U\), \(x \notin U\).

\textbf{Lemma 4.1.} A GTMS space \((X, g, m)\) is a \(T_0\)-space if and only if for any pair of distinct points \(x\) and \(y\) in \(X\), \(c_g(\{x\}) \neq c_g(\{y\})\) or \(c_m(\{x\}) \neq c_m(\{y\})\).

Proof. Suppose that \(X\) is a \(T_0\)-GTMS space and \(x, y \in X\) are distinct points. Then there exists a subset \(U\) which is either \(g\)-open or \(m\)-open such that \(x \in U\), \(y \notin U\) or \(y \in U\), \(x \notin U\). If \(x \in U\), \(y \notin U\) and \(U\) is \(g\)-open, then \(U \cap \{y\} = \emptyset\). Thus, \(x \notin c_g(\{y\})\), but \(x \in c_g(\{x\})\). Hence, \(c_g(\{x\}) \neq c_g(\{y\})\). If \(x \in U\), \(y \notin U\) and \(U\) is \(m\)-open, we can easily see that \(c_m(\{x\}) \neq c_m(\{y\})\).

Conversely, suppose that \(x, y \in X\) such that \(x \neq y\). If \(c_g(\{x\}) \neq c_g(\{y\})\), then there exists a point \(z \in X\) such that \(z \in c_g(\{x\})\) and \(z \notin c_g(\{y\})\). Thus, there exists a \(g\)-open subset \(U\) such that \(z \in U\) and \(U \cap \{y\} = \emptyset\), that is, \(y \notin U\). But \(x \in U\), since \(U \cap \{x\} \neq \emptyset\) because \(z \in c_g(\{x\})\). Now, if \(c_m(\{x\}) \neq c_m(\{y\})\), by the same way, we can see that there exists a \(m\)-open subset \(U\) such that \(x \in U\) and \(y \notin U\). Thus \(X\) is a \(T_0\)-GTMS Space. \qed

\textbf{Theorem 4.9.} Let \((X, g, m)\) be a GTMS space. If \(X\) is a \(T_{1/2}\)-GTMS space, then \(X\) is a \(T_0\)-GTMS space.
\textbf{Theorem 4.10.} Assume that $X$ is not $T_0$-GTMS space. By Lemma 4.1, there exists a pair of distinct points $x$ and $y$ in $X$ such that $c_m(\{x\}) = c_m(\{y\})$ and $c_g(\{x\}) = c_g(\{y\})$. Since $y \in c_m(\{y\})$, then $y \in c_m(\{x\})$. Thus $c_m(\{x\}) \neq \{\}$, that is; $\{x\}$ is not closed. If $X$ is $T_{1/2}$-GTMS space, then $\{x\}$ is open and hence $g$-open. Thus, $x \notin c_g(\{y\})$ and then $c_g(\{x\}) \neq c_g(\{y\})$, a contradiction. \hfill \Box

\textbf{Remark.} The converse of Theorem 4.9 is not true. We can be seen from the following example.

\textbf{Example 4.3.} Let $X = \{a, b, c\}$ with generalized topology $g = \{\emptyset, \{a, b\}\}$ and minimal structure $m = \{\emptyset, \{a\}, \{b\}, \{a, c\}, X\}$. Then $X$ is $T_0$-GTMS space but not $T_{1/2}$-GTMS space.

\textbf{Definition 4.3.} A GTMS space $(X, g, m)$ is called a $R_0$-GTMS space if $\{x\}$ is $G^*$-closed set for each $x \in X$.

\textbf{Theorem 4.10.} Let $(X, g, m)$ be a GTMS space. Then the following are equivalent.

1. $X$ is a $R_0$-GTMS space.
2. For each $x, y \in X$, if $x \notin c_g(\{y\})$, then $y \notin c_m(c_g(\{x\}))$ and if $x \notin c_m(\{y\})$, then $y \notin c_g(c_m(\{x\}))$.
3. For each $x, y \in X$, if $x \in c_m(c_g(\{y\}))$, then $y \in c_g(\{x\})$ and if $x \in c_g(c_m(\{y\}))$, then $y \in c_m(\{x\})$.

\textbf{Proof.} (1) $\implies$ (2) Suppose that $X$ is a $R_0$-GTMS space. Let $x, y \in X$ and $x \notin c_g(\{y\})$. So there exists a $g$-open subset $U$ such that $x \in U \subseteq X \setminus \{y\}$. Since $X$ is a $R_0$-GTMS space, then $\{x\}$ is $G^*$-closed and then $c_m(c_g(\{x\})) \subseteq U \cap \{y\} \neq \emptyset$. It follows that $y \notin c_m(c_g(\{x\}))$. If $x \notin c_m(\{y\})$, we can follow the similar manner to prove that $y \notin c_g(c_m(\{x\}))$.

(2) $\implies$ (3) It is obvious.

(3) $\implies$ (1) Let $\{x\} \subseteq U$, $U$ is $g$-open and $y \in c_m(c_g(\{x\}))$. Then $x \in c_g(\{y\})$. It shows that $U \cap \{y\} \neq \emptyset$ and hence $y \in U$. This proves that $c_m(c_g(\{x\})) \subseteq U$. Now, If $\{x\} \subseteq U$, $U$ is $m$-open, by the similar manner, we can prove that $c_g(c_m(\{x\})) \subseteq U$. Thus $\{x\}$ is $G^*$-closed. So $X$ is a $R_0$-GTMS space. \hfill \Box

\textbf{Theorem 4.11.} Let $(X, g, m)$ be a $R_0$-GTMS space. Then for each $x, y \in X$, $c_g(\{x\}) = c_g(\{y\})$ or $c_g(\{x\}) \cap c_g(\{y\}) = \emptyset$, also $c_m(\{x\}) = c_m(\{y\})$ or $c_m(\{x\}) \cap c_m(\{y\}) = \emptyset$.

\textbf{Proof.} Suppose that $(X, g, m)$ is a $R_0$-GTMS space and $x, y \in X$. If $c_g(\{y\}) \cap c_g(\{x\}) \neq \emptyset$, then there exists $z \in c_g(\{x\}) \cap c_g(\{y\})$. So $z \in c_g(\{x\})$. By Theorem 4.10, $x \in c_g(\{z\})$. It follows that $c_g(\{z\}) \subseteq c_g(\{x\})$ and $c_g(\{x\}) \subseteq c_g(\{z\})$. Consequently $c_g(\{z\}) = c_g(\{x\})$. In the same way,
\( c_g(\{z\}) = c_g(\{y\}) \). So \( c_g(\{x\}) = c_g(\{y\}) \). If \( c_m(\{x\}) \cap c_m(\{y\}) \neq \emptyset \), we can follow the same way to prove that \( c_m(\{x\}) = c_m(\{y\}) \).

\[ \square \]

**Theorem 4.12.** Let \((X, g, m)\) be a GTMS space. Then the following are equivalent.

1. \( X \) is a \( T_1 \)-GTMS space.
2. \( X \) is a \( T_0 \)-GTMS space and \( R_0 \)-GTMS space.

**Proof.**

(1) \( \implies \) (2) Suppose that \( X \) is a \( T_1 \)-GTMS space. By Theorem 4.8 and Theorem 4.9, \( X \) is \( T_0 \)-GTMS space. On the other hand, \( \{x\} \) is closed for each \( x \in X \), by using Theorem 2.2. Hence, \( \{x\} \) is \( G^* \)-closed, from Theorem 3.2. Thus \( X \) is \( R_0 \)-GTMS space.

(2) \( \implies \) (1) Let \( x, y \in X \) such that \( x \neq y \). We have the following two cases:

Case (i): There exists a \( g \)-open subset \( U \) such that \( x \in U \subseteq X \setminus \{y\} \), since \( X \) is a \( T_0 \)-GTMS space. Also, \( \{x\} \) is \( G^* \)-closed, since \( X \) is a \( R_0 \)-GTMS space. Hence \( y \notin c_m(c_g(\{x\})) \). Thus, \( \{x\} \) is closed.

Case (ii): There exists a \( m \)-open subset \( U \) such that \( x \in U \subseteq X \setminus \{y\} \). By the same way, we can prove that \( \{x\} \) is closed.

Thus \( X \) is a \( T_1 \)-GTMS space.

\[ \square \]

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**References**


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