Stabilities for Euler-Poisson Equations with Repulsive Forces in $\mathbb{R}^N$

Manwai Yuen

Department of Mathematics and Information Technology
The Hong Kong Institute of Education
10 Po Ling Road, Tai Po
New Territories, Hong Kong
nevetsyuen@hotmail.com

Copyright © 2013 Manwai Yuen. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

This article extends the previous paper (M.W. Yuen, Stabilities for Euler-Poisson Equations in Some Special Dimensions, J. Math. Anal. Appl. 344 (2008), 145–156) from the Euler-Poisson equations for attractive forces to the repulsive ones in $\mathbb{R}^N$ ($N \geq 2$). The similar stabilities of the system are studied. We also explain that it is impossible to have density collapsing solutions with compact support from classical solutions for a system with repulsive forces for $\gamma > 1$.

Mathematics Subject Classification: 35B30, 35B35, 35B40, 76Nxx

Keywords: Euler-Poisson Equations, Repulsive Forces, Stabilities, Frictional Damping, Second Inertia Function, Non-collapsing Solutions

1 Introduction

Semi-conductor models can be formulated by the isentropic Euler-Poisson equation with repulsive forces as follows:

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P + \beta \rho u &= \rho \nabla \Phi, \\
\Delta \Phi(t, x) &= \alpha(N) \rho,
\end{align*}
\]

(1)
where $\alpha(N)$ is a constant related to the unit ball in $\mathbb{R}^N$: $\alpha(1) = 2$; $\alpha(2) = 2\pi$; for $N \geq 3$,

$$\alpha(N) = N(N - 2)V(N) = N(N - 2)\frac{\pi^{N/2}}{\Gamma(N/2 + 1)}, \quad (2)$$

where $V(N)$ is the volume of the unit ball in $\mathbb{R}^N$ and $\Gamma$ is the Gamma function. As usual, $\rho = \rho(t, x)$ and $u = u(t, x) \in \mathbb{R}^N$ are the density and velocity respectively. $P = P(\rho)$ is the pressure and $\beta \geq 0$ is the frictional damping constant. In the preceding system, the self-repulsive potential field $\Phi = \Phi(t, x)$ is determined by the density $\rho$ through the Poisson equation. Equations (1)$_1$ and (1)$_2$ are compressible Euler equations with forcing terms. Equation (1)$_3$ is the Poisson equation through which the potential with repulsive forces is determined by the density distribution of the electrons themselves. Thus, we call system (1) the Euler-Poisson equations with repulsive forces. In this case, the equations can be viewed as comprising a semiconductor model. See [2] and [5] for details about the system. For some fixed $K \geq 0$, we have a $\gamma$-law on the pressure $P$, i.e.,

$$P(\rho) = K\rho^\gamma, \quad (3)$$

which is a common hypothesis. When $K = 0$, the pressureless system can be applied to models in plasma physics [1]. The constant $\gamma = c_P/c_v \geq 1$, where $c_P$ and $c_v$ are the specific heats per unit mass under constant pressure and constant volume, respectively, is the ratio of the specific heats. In addition, the fluid is isothermal if $\gamma = 1$.

The Poisson equation (1)$_3$ can be solved as

$$\Phi(t, x) = \int_{\mathbb{R}^N} G(x - y)\rho(t, y)dy, \quad (4)$$

where $G$ is the Green’s function for the Poisson equation in the $N$-dimensional spaces defined by

$$G(x) = \begin{cases} 
|x|, & N = 1; \\
\log |x|, & N = 2; \\
\frac{1}{|x|^{N-2}}, & N \geq 3.
\end{cases} \quad (5)$$

If we seek solutions in radial symmetry with the radius $r = \left(\sum_{i=1}^N x_i^2\right)^{1/2}$, the Poisson equation (1)$_3$ is transformed into

$$r^{N-1}\Phi_{rr} (t, x) + (N - 1) r^{N-2}\Phi_r = \alpha(N) \rho r^{N-1}, \quad (6)$$

$$\Phi_r = \frac{\alpha(N)}{r^{N-1}} \int_0^r \rho(t, s)s^{N-1}ds. \quad (7)$$
We can seek the radial symmetry solutions as follows:

\[ \rho(t, \vec{x}) = \rho(t, r) \text{ and } \vec{u} = \frac{\vec{x}}{r} V(t, r) = : \frac{\vec{x}}{r} V. \]  

(8)

By standard computation, the Euler-Poisson equations in radial symmetry can be written as follows:

\[
\begin{align*}
\rho_t + V \rho_r + \rho V_r + \frac{N-1}{r} \rho V &= 0, \\
\rho (V_t + V V_r) + P_r &= \rho \Phi_r (\rho).
\end{align*}
\]

(9)

Perthame discovered the blowup results for a 3-dimensional pressureless system with repulsive forces [9]. In short, all of the proceeding results rely on the solutions with radial symmetry:

\[
V_t + V V_r = \frac{a(N)}{r} \int_0^r \rho(t, s) s^{N-1} ds.
\]

(10)

The Emden ordinary differential equations are deduced on the boundary point of the solutions with compact support:

\[
\frac{D^2 R}{Dt^2} = M \frac{R^{N-1}}{R^N-1}, \quad R(0, R_0) = R_0 \geq 0, \quad \dot{R}(0, R_0) = 0,
\]

(11)

where \( \frac{dR}{dt} := V \) and \( M \) is the mass of the solutions along the characteristic curve. They show the blowup results for the \( C^1 \) solutions of system (9).

In 2011, Yuen ([13] and [14]) showed that the classical non-trivial solutions \((\rho, V)\) for the Euler or Euler-Poisson equations with repulsive forces in radial symmetry and with compact support in \([0, R]\), where \( R \) is a positive constant \((\rho(t, R) = 0, V(t, R) = 0)\) and the initial velocity such that

\[
H_0 = \int_0^R r^n V_0 dr > 0,
\]

(12)

blow up on or before the finite time \( T = 2R^{n+2}/(n(n + 1)H_0) \) for pressureless fluids \((K = 0)\) or \( \gamma > 1 \).

The systems with attractive forces were studied in [3], [6], [7], [8], [11] and [12]. This article extends [12] from the Euler-Poisson equations for attractive forces with or without frictional damping to the repulsive ones in \( R^N \) \((N \geq 2)\). The similar stabilities of the system are also studied.

In the third section, we exclude the possibility of collapsing solutions for this system. The non-existence of collapsing solutions can be shown by the simple following argument for the energy function:

\[
E(t) = \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma-1} P - \frac{1}{2} \rho \Phi \right) dx, \quad \text{for } \gamma > 1.
\]

(13)

**Theorem 1** For the classical solutions with compact support of the Euler-Poisson equations with repulsive force, (1), in \( R^N \) \((N \geq 2)\) with \( \gamma > 1 \) or without pressure \((K = 0)\), there is no instance where part of the density \( \rho(t, x) \) collapses to a point.
2 Stabilities

In this section, we study the stabilities of the Euler-Poisson equations with repulsive forces, (1), in $\mathbb{R}^N (N \geq 2)$. The total energy can be defined by

$$
E(t) = \int_\Omega \left( \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma - 1} P - \frac{1}{2} \rho \Phi \right) dx, \quad \text{for } \gamma > 1,
$$

$$
E(t) = \int_\Omega \left( \frac{1}{2} \rho |u|^2 - \frac{1}{2} \rho \Phi \right) dx, \quad \text{for without pressure.}
$$

(14)

For the system, we make the following energy estimate.

**Lemma 2** For the Euler-Poisson equations (1), suppose the solutions $(\rho, u)$ have compact support in $\Omega$. We have

$$
\dot{E}(t) = -\beta \int_\Omega \rho |u|^2 dx \leq 0.
$$

(15)

Sideris initially applied the second inertia function

$$
H(t) = \int_\Omega \rho(t, x) |x|^2 dx
$$

(16)

to study instability results for the Euler equations [10]. After that in [8], the instability result of the Euler-Poisson equations with attractive forces in radial symmetry was obtained for $\gamma \geq 4/3$ and $N = 3$. The corresponding cases in $\mathbb{R}^N$ with non-radial symmetry were studied in [3] and [12]. By the standard computation for energy method, the following lemma is clearly appropriate:

**Lemma 3** Consider that $(\rho, u)$ is a solution with compact support in $\Omega$ for the Euler-Poisson equations (1), with $\beta = 0$. We have

$$
\dot{H}(t) = 2 \int_\Omega \left[ (\rho |u|^2 + NP) dx - \frac{N-2}{2} \rho \Phi \right] dx, \quad \text{for } N \geq 3; \\
\dot{H}(t) = 2 \int_\Omega (\rho |u|^2 + 2P) dx + M^2, \quad \text{for } N = 2.
$$

(17)

By applying this lemma, we arrive at the following theorem.

**Theorem 4** Suppose that $(\rho, u)$ is a global classical solution in the Euler-Poisson equations (1), with $\gamma > 1$ without frictional damping ($\beta = 0$). We have

(1) for $N \geq 3$,

$$
\lim_{t \to \infty} \inf \frac{R(t)}{t} \geq \left[ \frac{\inf(2, N(\gamma - 1), N - 2)E}{M} \right]^{1/2};
$$

(18)
(2) for $N = 2$,

$$\lim_{t \to \infty} \inf \frac{R(t)}{t} \geq \sqrt{\frac{1}{2M}}; \quad (19)$$

(3) for $N \geq 2$,

$$\lim_{t \to \infty} \inf \frac{R(t)}{t} \geq \left[ \frac{NKM^{\gamma-1}}{|\Omega|^{(\gamma-1)}} \right]^{1/2}, \quad (20)$$

with $R(t) = \max_{x \in \Omega(t)} \{|x|\}$. Here,

$$M = \int_{\Omega} \rho(t, x) dx \quad (21)$$

is the total mass, which is constant for any classical solution, and $|\Omega|$ is the fixed volume of $\Omega$.

**Proof.** (1) For $N \geq 3$, we have the positive energy function $E \geq 0$. From Lemma 3, we have

$$\ddot{H}(t) = 2 \left\{ \int_{\Omega} [\rho |u|^2 + NP] \ dx - \frac{N-2}{2} \int_{\Omega} \rho \Phi \ dx \right\} \geq 2 \inf(2, N(\gamma - 1), N - 2)E \quad (22)$$

That is,

$$H(t) \geq H(0) + \dot{H}(0)t + \inf(2, N(\gamma - 1), N - 2)Et^2 \quad (23)$$

Further, we obtain

$$H(0) + \dot{H}(0)t + \inf(2, N(\gamma - 1), N - 2)Et^2 \leq H(t) \leq R(t)^2M \quad (24)$$

That is,

$$O\left(\frac{1}{t}\right) + \inf(2, N(\gamma - 1), N - 2)E \leq \frac{R(t)^2M}{t^2} \quad (25)$$

$$\lim_{t \to \infty} \inf \frac{R(t)}{t} \geq \left[ \frac{\inf(2, N(\gamma - 1), N - 2)E}{M} \right]^{1/2} \quad (26)$$

For $N = 2$, we have

$$\ddot{H}(t) = 2 \int_{\Omega} (\rho |u|^2 + 2P) \ dx + M^2 \geq M^2 \quad (27)$$
\[
\frac{M^2}{2} t^2 + C_0 t + C_1 \leq H(t) \leq R(t)^2 M, \quad (28)
\]

\[
\lim \inf_{t \to \infty} \frac{R(t)}{t} \geq \sqrt{\frac{1}{2M}}. \quad (29)
\]

For \(N \geq 2\), we obtain

\[
M = \int_{\Omega} \rho dx \leq \left( \int_{\Omega} \rho^\gamma dx \right)^{1/\gamma} |\Omega|^{(\gamma-1)/\gamma} \quad (30)
\]

and

\[
\ddot{H}(t) = 2 \left\{ \int_{\Omega} \left[ \rho \left| u \right|^2 + NP \right] dx - \frac{N - 2}{2} \int_{\Omega} \rho \Phi dx \right\} \geq 2 \int_{\Omega} NP dx. \quad (31)
\]

From the inequality in (30), it is clear that

\[
\ddot{H}(t) \geq 2NK |\Omega|^{1-\gamma} M^\gamma > 0, \quad (32)
\]

\[
H(0) + \dot{H}(0) t + NK |\Omega|^{1-\gamma} M^\gamma t^2 \leq H(t) \leq R(t)^2 M, \quad (33)
\]

\[
O\left(\frac{1}{t}\right) + NK |\Omega|^{1-\gamma} M^\gamma \leq \frac{R(t)^2 M}{t^2}. \quad (34)
\]

This gives

\[
\lim \inf_{t \to \infty} \frac{R(t)}{t} \geq \left[ \frac{NK M^{\gamma-1}}{|\Omega|^{(\gamma-1)}} \right]^{1/2}. \quad (35)
\]

This completes the proof. \(\blacksquare\)

By applying the method, in the preceding section to the system, (1), with frictional damping constant \((\beta > 0)\), we arrive at the following theorem.

**Theorem 5** Suppose that \((\rho, u)\) is a global classical solution with compact support in the system, (1), with frictional damping \((\beta > 0)\). For \(N \geq 2\), we have

\[
\lim \inf_{t \to \infty} \frac{R(t)}{t^{1/2}} \geq \left( \frac{2\beta NK M^{\gamma-1}}{|\Omega|^{(\gamma-1)}} \right)^{1/2}, \quad (36)
\]

with \(R(t) = \max_{x \in \Omega(t)} \{|x|\} \).
**Proof.** For \( N \geq 2 \), we have
\[
\dot{H}(t) = \int_{\Omega} 2x \rho u dx
\]  
(37)
and
\[
\ddot{H}(t) = 2 \int_{\Omega} x [-\nabla \cdot (\rho u \otimes u) - \nabla P + \rho \nabla \Phi - \beta \rho u] dx.
\]  
(38)
It can also be arranged as follows:
\[
\ddot{H}(t) = 2 \int_{\Omega} \left[ x - \nabla \cdot (\rho u \otimes u) - \nabla P + \rho \nabla \Phi \right] dx - \beta \dot{H}(t),
\]  
(39)
\[
\ddot{H}(t) + \frac{1}{\beta} \dot{H}(t) = 2 \left\{ \int_{\Omega} [\rho |u|^2 + NP] dx - \frac{N-2}{2} \int_{\Omega} \rho \Phi dx \right\}
\]  
\[
\geq 2 \int_{\Omega} NP dx \geq 2NK |\Omega|^{1-\gamma} M^\gamma > 0.
\]  
(40)
Therefore, we are able to obtain the following inequality:
\[
C_3 + C_4 e^{-\beta t} + 2\beta NK |\Omega|^{1-\gamma} M^\gamma t \leq H(t) \leq R(t)^2 M,
\]  
(41)
\[
O\left(\frac{1}{t}\right) + 2\beta NK |\Omega|^{1-\gamma} M^\gamma \leq \frac{R(t)^2 M}{t}.
\]  
(42)
This gives
\[
\liminf_{t \to \infty} \frac{R(t)}{t^{1/2}} \geq \left( \frac{2\beta NK M^{\gamma-1}}{|\Omega|^{(\gamma-1)}} \right)^{1/2}.
\]  
(43)
This completes the proof. \( \blacksquare \)

3 Non-existence of a Collapsing Solution

In this section, we reveal that it is impossible to have a density collapsing solution with compact support for the Euler-Poisson equations with repulsive forces.

We restate their energy for \( \gamma > 1 \) or the pressureless fluids as follows:
\[
E(t) = \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{1}{\gamma-1} P - \frac{1}{2} \rho \Phi \right) dx \geq - \int_{\Omega} \frac{1}{2} \rho \Phi dx.
\]  
(44)
When a $\delta$-shock exists for the density function $\rho(t, x)$, the potential energy function, with $N \geq 3$, becomes infinite:

$$ -\int_{\Omega} \rho \Phi dx = \int_{\Omega} \rho(t, x) \left( \int_{\Omega} \frac{\rho(t, y)}{|x - y|^{N-2}} dy \right) dx = \lim_{\epsilon \to 0^+} \int_{\Omega} \delta(t, x) \int_{\Omega} \frac{\delta(t, y)}{\epsilon^{N-2}} dy dx = +\infty. $$

(45)

With $N = 2$, the situation is similar:

$$ -\int_{\Omega} \rho \Phi dx = -\int_{\Omega} \delta(t, x) \int_{\Omega} \delta(t, y) \ln |x - y| dy dx = +\infty. $$

(46)

Therefore, the total energy of the $\delta$-shock density solutions must be infinite. However, for the classical solutions with compact support, the initial energy is finite. By the energy estimate in Lemma 2, we have

$$ E(t) \leq E(0). $$

(47)

If the total energy is finite, it is impossible to obtain the density collapsing solutions. Theorem 1 is clearly appropriate.

References


Received: September 9, 2013