Lower Bounds for Blow-up Time of Porous Medium Equation with Nonlinear Flux on Boundary

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Abstract

In this paper, we investigate the lower bounds for the blow-up time of the non-negative solutions of porous medium equation with Neumann boundary conditions. We find that the blow-up time are bounded below by $t^* \geq \int_0^\infty \frac{d \eta}{\Gamma(\eta)}$ for some computable function $\Gamma(\eta)$.

Mathematics Subject Classification: 35K65, 74H40

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1 Introduction

In this paper we are concerned with the lower bounds for the blow-up time of solutions of following porous medium equation with nonlinear flux on the boundary,

$$u_t = \Delta u^m - f(u), \quad (x, t) \in \Omega \times (0, T), \quad (1.1)$$

$$\frac{\partial u^m}{\partial \nu} = g(u), \quad (x, t) \in \partial \Omega \times (0, T), \quad (1.2)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (1.3)$$

where $m > 1$, the nonnegative initial value $u_0(x) \in C(\Omega) \cap L^\infty(\Omega)$, $\Omega$ is a bounded star-shaped region convex in two orthogonal directions in $\mathbb{R}^3$ with the sufficiently smooth

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boundary \( \partial \Omega \), \( \nu \) is the unit normal vector on \( \partial \Omega \), \( T \) is the blow-up time if blow-up occurs, or else \( T = \infty \).

There is an extensive literature on the bounds for the blow-up time of solutions of porous medium equation with non-linear sources and its linear counterpart—the heat equation, see [1, 2, 3, 4, 5], for other interesting results, see [6, 7, 8]. A variety of methods have used to determine the blow-up of solutions often indicate an upper bound for the blow-up time. In practical situations, we need to know the lower bound. In 2007, Payne and Schaefer [9] considered an initial-boundary value problem for the semilinear heat equation whose solution may blow up in finite time. They use two different methods to determine the lower bounds on blow-up time if blow-up occurs. In 2010, Payne, Philippin and Vernier [10] considered a semilinear heat equation with nonlinear boundary condition (\( m = 1 \) in (1.1)), and establish conditions on nonlinearities sufficient to guarantee that \( u(x, t) \) exists for all time \( t > 0 \) as well as conditions on data forcing the solution \( u(x, t) \) to blow up at some finite time \( t^* \). When \( N = 1 \), the blow-up phenomena for the solutions of the porous medium equation with nonlinear flux on boundary had also been studied by several authors [10, 11]. In our pervious paper [11], we had found that if the absorption is more powerful than the nonlinear boundary-flux, then the solutions of the problem (1.1)–(1.3) exist all the time on the bounded star-shaped region, on the other hand, if the nonlinear boundary-flux is more powerful, then the solutions of the problem (1.1)–(1.3) blow-up on a finite time. Moreover, we had given the upper-bound estimates for the blow-up time.

Our interesting in this paper is to find the low-bound estimates of the blow-up time for Problem (1.1)–(1.3). Here we have used some ideas of [10, 11, 4]. Our main result is the following theorem.

**Theorem 1.1.** Let \( u(x, t) \) be the nonnegative solution of problem (1.1)–(1.3),

\[
\begin{align*}
  f(\xi) &\geq k_1 \xi^p, \xi \geq 0, \quad (1.4) \\
  0 &\leq g(\xi) \leq k_2 \xi^{1+\frac{p}{2}}, \xi \geq 0, \quad (1.5)
\end{align*}
\]

for some nonnegative constants \( k_1, k_2, \) with \( n \geq 1 \) and \( p > 1 \). If \( u(x, t) \) becomes unbounded in the \( \phi \) measure at some finite time \( T \), then \( T \) is bounded below by \( T \geq \int_{\phi(0)}^{\infty} \frac{d\eta}{c_1 \eta + c_2 \eta^2 + c_3 \eta^3} \) for \( p = n + 1 \) or by \( T \geq \int_{\phi(0)}^{\infty} \frac{d\eta}{c_1 \eta + c_2 \eta^2 + c_3 \eta^3} = \frac{1}{2c_1} \ln \left(1 + \frac{c_1}{c_3} \phi^{-2}(0) \right) \) for \( p < n + 1 \).

**Remark 1.1.** Note that \( u(x, t) \) exists all time when \( p > n + 1 \), see[2].

Clearly if \( f(u) \) satisfy (1.4) with \( k_1 > 0 \), then if blow-up occurs it will at a time later than that when \( f(u) = 0 \).

## 2 Proofs of Results

**Proof.** To begin with the proofs, we define the auxiliary function

\[
\phi(t) = \int_{\Omega} u^{2n} dx, \quad (2.6)
\]
which satisfies a first order differential inequality of the form $\phi'(t) \leq \Gamma(\phi)$ for some computable function $\Gamma(\phi)$. Making use of (1.1)-(1.3), (1.4)-(1.5) and divergence theorem, we obtain

$$
\phi'(t) = 2n \int_{\Omega} u^{2n-1}[\Delta u^m - f(u)]dx
= 2n \int_{\Omega} u^{2n-1} \Delta u^m dx - 2n \int_{\Omega} u^{2n-1} f(u)dx
\leq 2nk_2 \int_{\partial\Omega} u^\frac{n}{2} ds - \frac{2m(2n-1)}{n} \int_{\Omega} u^{m-1} |\nabla u^n|^2 dx - 2nk_1 \int_{\Omega} u^{2n+p-1} dx \tag{2.7}
\leq 2nk_2 \left\{ \frac{3}{\rho_0} \int_{\Omega} u^\frac{n}{2} dx + \frac{5nd}{2\rho_0} \int_{\Omega} u^\frac{n}{2} |\nabla u| dx \right\}
- \frac{2m(2n-1)M}{n} \int_{\Omega} |\nabla u^n|^2 dx - 2nk_1 \int_{\Omega} u^{2n+p-1} dx
$$

with $\rho_0 = \min(x \cdot \nu)$, $d = \max_{\partial\Omega} |x|$ and $M$ is the lower bound of function $u^{m-1}$. Inserting the following inequalities

$$
\int_{\Omega} u^\frac{n}{2} dx \leq \left( \int_{\Omega} u^\frac{3n}{2} dx \int_{\Omega} u^\frac{2n}{2} dx \right)^\frac{1}{2} \leq \frac{1}{2} \int_{\Omega} u^3 dx + \frac{1}{2} \int_{\Omega} u^2 dx,
$$

$$
\int_{\Omega} u^\frac{n+1}{2} |\nabla u| dx = \frac{1}{n} \int_{\Omega} u^\frac{n}{2} |\nabla u^n| dx \frac{1}{n} \left( \int_{\Omega} u^\frac{3n}{2} dx \int_{\Omega} |\nabla u^n|^2 dx \right)^\frac{1}{2}
 \leq \frac{1}{2\mu} \int_{\Omega} u^3 dx + \frac{\mu}{2n^2} \int_{\Omega} |\nabla u^n|^2 dx,
$$

and $\int_{\Omega} u^{2n+p-1} dx \geq |\Omega|^\frac{1-p}{2n} \phi^{1+\frac{n-1}{2n}}$ in (2.7), we obtain

$$
\phi'(t) \leq \frac{6nk_2}{\rho_0} \left( \frac{1}{2} \int_{\Omega} u^3 dx + \frac{1}{2} \int_{\Omega} u^2 dx \right) + \frac{5n^2dk_2}{\rho_0} \left( \frac{1}{2\mu} \int_{\Omega} u^3 dx + \frac{\mu}{2n^2} \int_{\Omega} |\nabla u^n|^2 dx \right)
- \frac{2m(2n-1)M}{n} \int_{\Omega} |\nabla u^n|^2 dx - 2nk_1 |\Omega|^\frac{1-p}{2n} \phi^{1+\frac{n-1}{2n}}
= \left( \frac{3nk_2}{\rho_0} + \frac{5n^2dk_2}{2\rho_0} \right) \int_{\Omega} u^3 dx + \frac{3nk_2}{\rho_0} \int_{\Omega} u^2 dx
+ \left( \frac{5d\mu k_2}{2\rho_0} - \frac{2m(2n-1)M}{n} \right) \int_{\Omega} |\nabla u^n|^2 dx - 2nk_1 |\Omega|^\frac{1-p}{2n} \phi^{1+\frac{n-1}{2n}}
= \frac{3nk_2}{\rho_0} \phi + \frac{nk_2}{\rho_0} \left( 3 + \frac{5nd}{2\mu} \right) \int_{\Omega} u^3 dx
+ \left( \frac{5\mu k_2}{2\rho_0} - \frac{2m(2n-1)M}{n} \right) \int_{\Omega} |\nabla u^n|^2 dx - 2nk_1 |\Omega|^\frac{1-p}{2n} \phi^{1+\frac{n-1}{2n}} \tag{2.8}
$$
for \( \mu > 0 \) to be chosen. Next we use of the following Sobolev type inequality deduced by Payne and Shafer in [18],

\[
\int_{\Omega} u^{3n} dx \leq \frac{\sqrt{2}}{3^3} \left\{ \left( \frac{3}{2\rho_0} \right)^{\frac{3}{4}} \phi^{\frac{3}{4}} + \left( \frac{d}{\rho_0} + 1 \right)^{\frac{3}{4}} \left( \int_{\Omega} |\nabla u^n|^2 dx \right)^{\frac{3}{4}} \right\}
\]

(2.9)

valid for a bounded star-shaped region \( \Omega \) in \( \mathbb{R}^3 \) assumed to be convex in two orthogonal directions and for \( \lambda > 0 \). Combining (2.8) and (2.9), we obtain

\[
\phi'(t) \leq \frac{6n^2k_2}{\rho_0} \left( \frac{1}{2} \int_{\Omega} u^{3n} dx + \frac{1}{2} \int_{\Omega} u^{2n} dx \right) + \frac{5n^2d^2k_2}{\rho_0} \left( \frac{1}{2\mu} \int_{\Omega} u^{3n} dx + \frac{\mu}{2n^2} \int_{\Omega} |\nabla u^n|^2 dx \right) - 2m(2n-1)M \int_{\Omega} |\nabla u^n|^2 dx - 2nk_1|\Omega|^{\frac{1-p}{2n}} \phi^{1+\frac{p-1}{2n}}
\]

(2.10)

\[
= \left( \frac{3nk_2}{\rho_0} + \frac{5n^2d^2k_2}{2\rho_0} \right) \int_{\Omega} u^{3n} dx + \frac{3nk_2}{\rho_0} \int_{\Omega} u^{2n} dx + \frac{5n^2d^2k_2}{2\rho_0} \int_{\Omega} \frac{2m(2n-1)M}{n} \int_{\Omega} |\nabla u^n|^2 dx - 2nk_1|\Omega|^{\frac{1-p}{2n}} \phi^{1+\frac{p-1}{2n}}
\]

\[
\leq \frac{3nk_2}{\rho_0} \phi + \frac{n^2k_2}{\rho_0} \left( \frac{3}{2\mu} \right) \left( \frac{5n^2d^2k_2}{\rho_0} \right) \left( \frac{\phi^{\frac{3}{4}}}{3^3} \phi^{\frac{3}{4}} + \left( \frac{d}{\rho_0} + 1 \right)^{\frac{3}{4}} \left( \int_{\Omega} |\nabla u^n|^2 dx \right)^{\frac{3}{4}} \right) + \frac{5n^2d^2k_2}{2\rho_0} \int_{\Omega} \frac{2m(2n-1)M}{n} \int_{\Omega} |\nabla u^n|^2 dx - 2nk_1|\Omega|^{\frac{1-p}{2n}} \phi^{1+\frac{p-1}{2n}}
\]

(2.10)

for nonnegative constants \( c_i \) given by

\[
c_1 = \frac{3nk_2}{\rho_0}, \quad c_2 = \frac{3^n k_2}{2\rho_0^2} \left( \frac{3}{2\mu} \right) \left( 3 + \frac{5n^2d}{\rho_0} \right), \quad c_3 = \frac{n^2k_2}{3^3 2^4 \rho_0^2 \lambda^3} \left( 3 + \frac{5n^2d}{\rho_0} \right) \left( \frac{d}{\rho_0} + 1 \right)^{\frac{3}{2}}
\]

\[
c_4 = \frac{3^n k_2 n \lambda}{2^2 \rho_0^2} \left( 3 + \frac{5n^2d}{\rho_0} \right) \left( \frac{d}{\rho_0} + 1 \right)^{\frac{3}{2}} + \frac{5n^2d^2k_2}{2\rho_0} \int_{\Omega} \frac{2m(2n-1)M}{n} \int_{\Omega} |\nabla u^n|^2 dx - 2nk_1|\Omega|^{\frac{1-p}{2n}} \phi^{1+\frac{p-1}{2n}}
\]

For \( \mu > 0 \) small enough, we then choose \( \lambda > 0 \) such that \( c_4 = 0 \) and we can obtain

\[
\phi'(t) \leq c_1 \phi + c_2 \phi^{\frac{3}{4}} + c_3 \phi^{\frac{3}{2}} - 2nk_1|\Omega|^{\frac{1-p}{2n}} \phi^{1+\frac{p-1}{2n}}.
\]

(2.11)
In the particular case \( p = n + 1 \), (2.11) then reduces to
\[
\phi'(t) \leq c_1 \phi + \tilde{c}_2 \phi^\frac{3}{2} + c_3 \phi^3
\]  
\[
(2.12)
\]
with \( \tilde{c}_2 = c_2 - 2nk_1|\Omega|^{-\frac{1}{2}} \geq 0 \). From (2.12) we obtain that \( T \) is bounded below by
\[
T \geq \int_{\phi(0)}^{\infty} \frac{d\eta}{c_1 \eta + \tilde{c}_2 \eta^{\frac{3}{2}} + c_3 \eta^3}.
\]  
\[
(2.13)
\]
If \( p < n + 1 \) we eliminate the last term in (2.11) by using the following inequality
\[
\phi^3 = \left( \varepsilon \phi^{1+\frac{n-1}{2n}} \right)^{\frac{3n}{4n+1-p}} \left( \phi^3 \varepsilon^{\frac{3n}{p-n-1}} \right)^{\frac{n+1-p}{4n+1-p}}
\]
\[
\leq \frac{3n\varepsilon}{4n + 1 - p} \phi^{1+\frac{n-1}{2n}} + \frac{n+1-p}{4n + 1 - p} \varepsilon^{\frac{3n}{p-n-1}} \phi^3
\]  
\[
(2.14)
\]
valid for \( \varepsilon > 0 \), and let \( \varepsilon \) such that \( \frac{3n}{4n + 1 - p} c_2 - 2k_1|\Omega|^{\frac{1-p}{2m}} = 0 \). By the argument above we obtain \( \phi'(t) \leq c_1 \phi + \tilde{c}_3 \phi^3 \) with \( \tilde{c}_3 = c_3 + c_2 \frac{n+1-p}{4n+1-p} \varepsilon^{\frac{3n}{p-n-1}} \), from which we obtain that \( T \) is bounded below by
\[
T \geq \int_{\phi(0)}^{\infty} \frac{d\eta}{c_1 \eta + \tilde{c}_3 \eta^3} = \frac{1}{2c_1} \ln \left( 1 + \frac{c_1}{\tilde{c}_3} \phi^{-2}(0) \right).
\]

So we complete the proof of our main theorem. \( \square \)

References


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