A Note on the Generalized Twisted Tangent Polynomials

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Abstract

In this paper we introduce the generalized twisted tangent numbers $T_{n,\chi,w}$ and polynomials $T_{n,\chi,w}(x)$. Some interesting results and relationships are obtained.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: tangent numbers and polynomials, twisted tangent numbers and tangent polynomials, generalized twisted tangent numbers and polynomials

1 Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of $p$-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $\nu(p) = p^{-\nu_p(p)} = p^{-1}$. When one talks of extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

For $g \in UD(\mathbb{Z}_p) = \{g|g: \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}$,
the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(g) = \int_X g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x,$$

see [4, 5]. (1.1)

If we take $g_n(x) = g(x + n)$ in (1.1), then we see that

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l).$$

(1.2)

Let $T_p = \bigcup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{ w | w^{p^N} = 1 \}$ is the cyclic group of order $p^N$. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto w^x$.

Ryoo [11] introduced the twisted tangent polynomials $T_{n,w}(x)$. The twisted tangent numbers $T_{n,w}$ are defined by the generating function:

$$F_w(t) = 2 \sum_{a=0}^{w-1} \chi(a) (-1)^a w^a e^{2at} = \sum_{n=0}^\infty T_{n,w} \frac{t^n}{n!} \quad (|t| < \frac{\pi}{2}), \text{ cf. } [6,10]$$

(2.1)

where we use the technique method notation by replacing $(T_w)^n$ by $T_{n,w}(n \geq 0)$ symbolically. We consider the tangent polynomials $T_{n,w}(x)$ as follows:

$$F_w(x,t) = \left( \frac{2}{w e^{2t} + 1} \right) e^{xt} = \sum_{n=0}^\infty T_{n,w}(x) \frac{t^n}{n!}. \quad (1.4)$$

Note that $T_{n,w}(x) = \sum_{k=0}^n (-1)^k T_{k,w} x^{n-k}$. In the special case $x = 0$, we define $T_{n,w}(0) = T_{n,w}$.

The purpose of this paper is to construct the generalized twisted tangent polynomials $T_{n,\chi,w}(x)$ attached to $\chi$ and derive a new $l$-series which interpolates the generalized twisted tangent polynomials $T_{n,\chi,w}(x)$.

### 2 Generalized twisted tangent polynomials

In this section, our goal is to give generating functions of the generalized twisted tangent numbers and polynomials. These numbers will be used to prove the analytic continuation of the $l$-series. Let $w$ be the $p^N$-th root of unity. Let $\chi$ be Dirichlet’s character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. Then the generalized twisted tangent numbers associated with $\chi$, $T_{n,\chi,w}$, are defined by the following generating function

$$F_{\chi,w}(t) = \frac{2 \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a e^{2at}}{w^d e^{2t} + 1} = \sum_{n=0}^\infty T_{n,\chi,w} \frac{t^n}{n!}. \quad (2.1)$$
We now consider the generalized twisted tangent polynomials associated with \( \chi, T_{n,\chi,w}(x) \), are also defined by
\[
F_{\chi,w}(x, t) = \left( 2 \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a e^{2at} / e^{2dt} + 1 \right) e^{xt} = \sum_{n=0}^{\infty} T_{n,\chi,w}(x) \frac{t^n}{n!}. \tag{2.2}
\]

When \( \chi = \chi^0 \), above (2.1) and (2.2) will become the corresponding definitions of the twisted tangent numbers \( T_{n,w} \) and polynomials \( T_{n,w}(x) \).

Since
\[
2 \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a e^{2at} / e^{2dt} + 1
= \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a \left( \frac{2e^{(\frac{2a+x}{d})dt}}{e^{2dt} + 1} \right)
= \sum_{m=0}^{\infty} \left( \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a T_{m,w} \left( \frac{2a+x}{d} \right) \right) \frac{t^m}{m!},
\]
we have the following theorem.

**Theorem 2.1** Let \( \chi \) be Dirichlet’s character with conductor \( d \in \mathbb{N} \) with \( d \equiv 1(\text{mod}2) \). Then we have

\[
(1) \quad T_{n,\chi,w}(x) = d^m \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a T_{m,w} \left( \frac{2a+x}{d} \right),
\]

\[
(2) \quad T_{n,\chi,w} = d^m \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a T_{m,w} \left( \frac{2a}{d} \right),
\]

\[
(3) \quad T_{n,\chi,w}(x) = \sum_{l=0}^{n} \binom{n}{l} T_{l,\chi,w} x^{n-l}.
\]

For \( n \in \mathbb{N} \) with \( n \equiv 0(\text{mod}2) \), we have
\[
-2 \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a e^{2at} / e^{2dt} + 1 w^{nd} e^{2ndt} + 2 \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a e^{2at} / e^{2dt} + 1
= \sum_{m=0}^{\infty} \left( 2 \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a (2a)^m \right) \frac{t^m}{m!}
\]

By comparing coefficients of \( \frac{t^m}{m!} \) in the above equation, we have the following theorem:
Theorem 2.2 Let $\chi$ be Dirichlet’s character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, $n$ a positive even integer, and $m \in \mathbb{N}$. Then we have

$$2 \sum_{a=0}^{nd-1} \chi(a)(-1)^a w^a (2a)^m = -w^{nd} T_{m,\chi,w}(2nd) + T_{m,\chi,w}.$$ 

Next, we introduce the $l$-series and two variable $l$-series.

**Definition 2.3** For $s \in \mathbb{C}$ with $\text{Re}(s) > 0$, define two variable $l$-series as

$$l_w(s, x|\chi) = 2 \sum_{m=0}^{\infty} \frac{(-1)^m \chi(m) w^m}{(2m + x)^s}.$$ 

By using (2.2), we easily see that

$$F_{\chi,w}(x, t) = \left. \frac{d}{dt} \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a e^{2at} \right|_{t=0} = \frac{2d}{w d e^{2dt} + 1} e^{xt}$$

$$= 2 \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a e^{(2a + x)t} \sum_{l=0}^{\infty} (-1)^l w^{ld} e^{2dt}$$

$$= 2 \sum_{a=0}^{d-1} \sum_{l=0}^{\infty} \chi(a)(-1)^{a + dl} w^{a + dl} e^{(2a + x + 2dt)t}$$

$$= 2 \sum_{m=0}^{\infty} \chi(m)(-1)^m w^m e^{(2m + x)t}.$$ 

Then we have

$$\left. \left(\frac{d}{dt}\right)^k F_{\chi,w}(x, t) \right|_{t=0} = 2 \sum_{n=0}^{\infty} \chi(n)(-1)^n w^n (2n + x)^k, \quad (2.3)$$

and

$$\left. \left(\frac{d}{dt}\right)^k \left( \sum_{n=0}^{\infty} T_{n,\chi,w}(x) \frac{t^n}{n!} \right) \right|_{t=0} = T_{k,\chi,w}(x), \quad \text{for } k \in \mathbb{N}. \quad (2.4)$$

By (2.3), (2.4), we have the following theorem.

**Theorem 2.4** For any positive integer $k$, we have

$$T_{k,\chi,w}(x) = l_w(-k, x|\chi).$$

**Definition 2.5** For $s \in \mathbb{C}$ with $\text{Re}(s) > 0$, define $l$-series as

$$l_w(s |\chi) = 2 \sum_{m=1}^{\infty} \frac{(-1)^m \chi(m) w^m}{(2m)^s}.$$ 

By simple calculation, we have the following theorem.

**Theorem 2.6** For any positive integer $k$, we have

$$l_w(-k |\chi) = T_{k,\chi,w}.$$
3 \quad \textbf{Witt-type formulae on } \mathbb{Z}_p \text{ in } p\text{-adic number field}

Our primary aim in this section is to obtain the Witt-type formulae of the generalized twisted tangent numbers \( T_{n,\chi,w} \) and polynomials \( T_{n,\chi,w}(x) \) attached to \( \chi \). Let \( \chi \) be the primitive Dirichlet character with conductor \( d \in \mathbb{N} \) with \( d \equiv 1(\text{mod}2) \) and \( w \in T_p \). Let \( g(y) = \chi(y)\phi_w(y)e^{(2y+x)t} \). By (1.1), we derive

\[
I_1 (\chi(y)\phi_w(y)e^{(2y+x)t}) = \int_X \chi(y)\phi_w(y)e^{(2y+x)t}d\mu_{-1}(y)
= \left( \frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^aw^ae^{2at}}{w^de^{2dt} + 1} \right)e^{xt}
= \sum_{n=0}^{\infty} T_{n,\chi,w}(x)\frac{t^n}{n!}.
\]

By using Taylor series of \( e^{(2y+x)t} \) in the above equation (3.1), we obtain

\[
\sum_{n=0}^{\infty} \left( \int_X \chi(y)\phi_w(y)(2y + x)^nd\mu_{-1}(y) \right)\frac{t^n}{n!} = \sum_{n=0}^{\infty} T_{n,\chi,w}(x)\frac{t^n}{n!}.
\]

By comparing coefficients of \( \frac{t^n}{n!} \) in the above equation, we have the Witt formula for the generalized twisted Tangent polynomials attached to \( \chi \) as follows:

\textbf{Theorem 3.1} For positive integers \( n \) and \( w \in T_p \), we have

\[
T_{n,\chi,w}(x) = \int_X \chi(y)\phi_w(y)(2y + x)^nd\mu_{-1}(y). \tag{3.2}
\]

Observe that for \( x = 0 \), the equation (3.2) reduces to (3.3).

\textbf{Corollary 3.2} For positive integers \( n \) and \( w \in T_p \), we have

\[
T_{n,\chi,w} = \int_X \chi(y)\phi_w(y)(2y)^nd\mu_{-1}(y). \tag{3.3}
\]

By (3.1) and (1.2), we have the following theorem:

\textbf{Theorem 3.3} For positive integers \( n \) and \( w \in T_p \), we have

\[
w^{nd}T_{m,\chi,w}(2nd) - (-1)^nT_{m,\chi,w} = 2^{m+1}\sum_{l=0}^{nd-1}(-1)^{n-1-l}\chi(l)w^lm^n.
\]
References


Received: September 5, 2013