Some Interpolation Problem for an \(F\)-space

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Abstract

Some interpolation problem is studied for an \(F\)-space. This generalizes a known result of an \(\ell^1\)-interpolation problem for a Banach space. Then there is an application for Smirnov class and Arens-Hardy space.

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§1 Introduction

Let \(B\) be an \(F\)-space with an invariant metric \(d_B\) and \(B^*\) its dual space. Throughout this paper, we assume \(d_B\) satisfies the following conditions : (1) When \(\alpha, \beta \in \mathbb{C}\) and \(f\) is nonzero in \(B\), \(d_B(\alpha f, 0) > d_B(\beta f, 0)\) if and only if \(|\alpha| > |\beta|\). (2) \(d_B(\alpha f, 0)\) is continuous with respect to \(\alpha \in \mathbb{C}\) for each \(f\) in \(B\).

We assume that \((\phi_n)\) is an infinite sequence of distinct points in \(B^*\). Let \(\ell\) be a sequence space of \((w_n)\) where \(w_n \in \mathbb{C}\). A sequence \((\phi_n)\) is called \(\ell\)-interpolating if for every sequence \((w_n)\) in \(\ell\) there exists an element \(f\) in \(B\) such that \(\phi_n(f) = w_n\) for all \(n\). For \((\phi_n)\) in \(B^*\) put

\[ J = \{ f \in B : f = 0 \text{ on } (\phi_k) \}, \]

\[ J_n = \{ f \in B : f = 0 \text{ on } (\phi_k)_{k \neq n} \}, \]
and 

\[ \rho_n = \sup\{|\phi_n(f)| : f \in J_n, d_B(f, 0) \leq 1\} \]

Then \( 0 \leq \rho_n \leq \infty \). In general, \( \rho_n > 0 \) if and only if \( J_n \neq J \). Hence \( 0 < \rho_n \leq \infty \) if and only if there exists an element \( f_n \) in \( B \) such that \( \phi_k(f_n) = \delta_{kn} \). In this paper, we assume that \( \rho_n \neq 0 \) for all \( n \) and so \( J_n = \langle f_n \rangle + J \). For each \( \alpha \) in \( \mathbb{C} \), put

\[ \varepsilon^B_n(\alpha) = \frac{d_B(\alpha f_n + J, 0)}{d_B(f_n + J, 0)} \quad (n = 1, 2, \ldots). \]

Then \( \varepsilon^B_n(1) = 1 \) and \( \varepsilon^B_n(0) = 0 \). We define \( \varepsilon^B_n(\infty) \) as the following : \( \varepsilon^B_n(\infty) = \sup_{\alpha \in \mathbb{C}} \varepsilon^B_n(\alpha) \).

For \( (\phi_n) \) in \( B^* \), put

\[ \ell_B(\phi_n) = \{ (w_n) : \sum_{n=1}^{\infty} \varepsilon^B_n(w_n) < \infty \} \]

We define a metric \( D_{\ell_B} \) on \( \ell_B(\phi_n) \) as \( D_{\ell_B}(u, v) = \sum_{n=1}^{\infty} \varepsilon^B_n(u_n - v_n) \) where \( u = (u_n) \) and \( v = (v_n) \). Then by the definition of \( \varepsilon^B_n \), \( D_{\ell_B} \) is an invariant metric and \( D_{\ell_B}((\delta_{kn}), 0) = 1 \) for any \( k \geq 1 \). If \( 0 < p \leq 1 \) and \( d_B(\alpha f, 0) = |\alpha|^p d(f, 0) \) \( (f \in B) \) then \( \ell_B(\phi_n) = \ell^p \). If \( B \) is a Banach space then \( \ell_B(\phi_n) = \ell^1 \).

In this paper, we give a necessary and sufficient condition for an \( \ell_B(\phi_n) \)-interpolation problem. In the previous paper [3] and [4], the problem have been solved. Unfortunately it has not solved when \( \ell_B(\phi_n) \neq \ell^p \) \( (0 < p \leq 1) \).

§2 A general theorem for an \( F \)-space

In this section, we give a necessary and sufficient condition for \( \ell_B(\phi_n) \)-interpolating sequence when \( \ell_B(\phi_n) \) is an \( F \)-space. See [5, p8] about the definition of boundedness in a topological vector space.

**Lemma 1.** If \( (\phi_n) \) is an \( \ell_B(\phi_n) \)-interpolating sequence then \( \sup_n d_B(f_n + J, 0) < \infty \).

**Proof.** Put \( S = (\phi_n) \). For \( (w_n) \in \ell_B(\phi_n) \), put

\[ T(w_n) = \sum_{n=1}^{\infty} w_n(f_n \mid S) \]

then by the hypothesis there exists \( f \) in \( B \) such that \( T(w_n) = f \mid S \). We put the metric of \( B/J \) on \( B \mid S \). By the closed graph theorem, \( T \) is bounded from \( \ell_B(\phi_n) \) to \( B \mid S \).
Set \( E = \{ (\delta_{kn}) \in \ell_B(\phi_n) : k = 1, 2, \ldots \} \subset \ell_B(\phi_n) \). Then \( E \) is a bounded set in \( \ell_B(\phi_n) \) because \( D\ell_B(\delta_{kn}),0) = 1 \). Hence \( TE \) is also a bounded set in \( B \mid_S \) because \( T \) is bounded. Thus \( \{ f_k \mid S : k = 1, 2, \ldots \} \) is a bounded set and so \( \sup_k d_B(f_k + J,0) < \infty \).

**Lemma 2.** If \( \sup_n d_B(f_n + J,0) < \infty \) then \( (\phi_n) \) is an \( \ell_B(\phi_n) \)-interpolating sequence.

Proof. Suppose \( \gamma = \sup_n d_B(f_n + J,0) < \infty \) and \( (w_n) \in \ell_B(\phi_n) \). For each \( n \) there exists \( g_n \) in \( J \) such that \( d_B(w_n(f_n + g_n),0) \leq d_B(w_nf_n + J,0) + 2^{1-n} \).

Put \( f^{(\ell)} = \sum_{n=1}^{\ell} w_n(f_n + g_n) \) then \( f^{(\ell)} \in B \) and for \( \ell \geq k + 1 \)

\[
d_B(f^{(\ell)} - f^{(k)},0) \leq \sum_{n=k+1}^{\ell} d_B(w_n(f_n + g_n),0) \leq \sum_{n=k+1}^{\ell} \{ d_B(w_nf_n + J,0) + 2^{1-n} \} = \sum_{n=k+1}^{\ell} \varepsilon_n^B(w_n) d_B(f_n + J,0) + \sum_{n=k+1}^{\ell} 2^{1-n} \leq \gamma \sum_{n=k+1}^{\ell} \varepsilon_n^B(w_n) + 2(2^{-k} - 2^{-\ell}).
\]

This shows that \( \{ f^{(\ell)} \} \) is a Cauchy sequence in \( B \) and so \( f = \lim_{\ell \to \infty} f^{(\ell)} \) belongs to \( B \) because \( B \) is an \( F \)-space. Then for each \( n \),

\[
\phi_n(f) = \lim_{\ell \to \infty} \phi_n(f^{(\ell)}) = w_n\phi_n(f_n) = w_n
\]

because \( \phi_n(f_k) = \delta_{nk} \). This shows \( (\phi_n) \) is an \( \ell_B(\phi_n) \)-interpolating sequence.

**Lemma 3.** Suppose \( \ell_B(\phi_n) \) is an \( F \)-space. \( (\phi_n) \) is an \( \ell_B(\phi_n) \)-interpolating sequence if and only if \( \sup_n d_B(f_n + J,0) < \infty \).

Proof. Lemmas 1 and 2 show this lemma.

For each \( n \), put

\[
\kappa_n = \sup_{f \in J_n} d_B(f,0).
\]

In general, \( 0 < \kappa_n \leq \infty \). When \( \kappa_n = \infty \), \( \rho_n < \infty \). When \( \kappa_n < \infty \), two cases may happen. That is, if \( \kappa_n \leq 1 \) then \( \rho_n = \infty \) and if \( \kappa_n > 1 \) then \( \rho_n < \infty \). If \( \kappa_n = \infty \) then by the hypothesis (2) on \( d_B \) there exists \( 0 < \alpha_n < \infty \) such that \( d_B(\alpha_n f_n + J,0) = 1 \).

**Lemma 4.** If \( \kappa_n > 1 \) then \( d_B(f_n + J,0) = 1/\varepsilon_n^B(\rho_n) \). If \( \kappa_n \leq 1 \) then \( d_B(f_n + J,0) = \kappa_n/\varepsilon_n^B(\rho_n) \).

Proof. If \( \kappa_n > 1 \) then \( \rho_n < \infty \). By the definition of \( \rho_n \) and the hypothesis (2) on \( d_B \), there exist \( \alpha_{nj} \) in \( \mathbb{C} \) such that \( d_B(\alpha_{nj}f_n + J,0) \leq 1 \) and \( |\alpha_{nj}| \nearrow \rho_n \) \( (j \to \infty) \). By \( \kappa_n > 1 \) and the hypothesis (2) on \( d_B \), there exists \( \alpha_n \in \mathbb{C} \)
such that \( d_B(\alpha_n f_n + J, 0) = 1 \). By the hypothesis (1) on \( d_B \) and the definition of \( \rho_n, |\alpha_n| \leq |\alpha| \leq \rho_n \) and so \( |\alpha| = \rho_n \). Hence \( d_B(f_n + J, 0) = 1/\varepsilon_n^B(\rho_n) \).

Suppose \( \kappa_n \leq 1 \). Then \( \rho_n = \infty \) and so by definition \( \varepsilon_n^B(\rho_n) = \sup_{\alpha \in \mathbb{C}} d(\alpha f_n + J, 0)/d(f_n + J, 0) \). Since \( \sup_{\alpha \in \mathbb{C}} d(\alpha f_n + J, 0) = \sup_{f \in J_n} d(f, 0) = \kappa_n, \varepsilon_n^B(\rho_n) = \kappa_n \).

**Lemma 5.** A sequence space \( \ell_B(\phi_n) \) is an \( F \)-space.

Proof. Let \( w^k = (w^k_n) \) be a Cauchy sequence in \( \ell_B(\phi_n) \). Then for any \( n \)

\[
\varepsilon_n^B(w^k_n - w^\ell_n) \leq \sum_{j=1}^{\infty} \varepsilon_j^B(w^k_j - w^\ell_j) = D_{\ell_B}(w^k - w^\ell) \to 0 \quad (k, \ell \to \infty).
\]

Then there exists \( w_n \in \mathbb{C} \) such that \( \lim_{k \to \infty} w^k_n = w_n \) because \( d_B \) satisfies the condition (1) and (2) in Introduction. Put \( w = (w_n) \) then we can prove that \( w \in \ell_B(\phi_n) \) and \( D_{\ell_B}(w^k - w) \to 0 \) as \( k \to \infty \) by a familiar argument.

**Theorem 1.** Then \((\phi_n)\) is an \( \ell_B(\phi_n) \)-interpolating sequence if and only if \( \inf_n \varepsilon_n^B(\rho_n) > 0 \).

Proof. It is a result of Lemmas 3, 4 and 5.

**Corollary 1.** Suppose \( d_B(\alpha f, 0) = |\alpha|^p d_B(f, 0) \) for some \( 0 < p \leq 1 \). Then \( \varepsilon_n^B(\rho_n) = \rho_n^p \) for \( n = 1, 2, \ldots \). Hence \((\phi_n)\) is an \( \ell^p \)-interpolating sequence if and only if \( \inf_n \rho_n > 0 \).

Proof. If \( d_B(\alpha f, 0) = |\alpha|^p d_B(f, 0) \), then \( \varepsilon_n^B(\alpha) = |\alpha|^p \) and so \( \ell_B(\phi_n) = \ell^p \). Hence by Theorem 1, \((\phi_n)\) is an \( \ell^p \)-interpolating sequence if and only if \( \inf_n \rho_n^p > 0 \).

§3 Concrete examples

Let \( D \) be the open unit disc in \( \mathbb{C} \) and \( H(D) \) the set of all holomorphic functions on \( D \). For \((a_n) \) in \( D \) with \( \sum_{n=1}^{\infty} (1 - |a_n|) < \infty \), put \( \phi_n(f) = f(a_n) \) \((f \in H(D))\). For \( 1 \leq n < \infty \), put

\[
B_n(z) = \prod_{j \neq n,j=1}^{\infty} \frac{\bar{a}_j z - a_j}{a_j - \bar{a}_j z} \text{ and } f_n(z) = B_n(z)/B_n(a_n)
\]

then \( \phi_j(f_n) = f_n(a_j) = \delta_{nj} \) and \( f_n \) belongs to \( H(D) \) and it is bounded.

For \( 0 < p \leq \infty \), \( H^p(D) \) denotes the usual Hardy space on \( D \) and put

\[
\|f\|_p^p = \int_{0}^{2\pi} |f(e^{i\theta})|^p d\theta/2\pi \quad (f \in H^p(D)).
\]
$N_+(D)$ denotes the Smirnov class on $D$ and then

$$d_{N_+}(f, g) = \int_0^{2\pi} \log(1 + |f(e^{i\theta}) - g(e^{i\theta})|)d\theta/2\pi$$

is an invariant metric on $N_+(D)$ (see [6]). $H_\omega(D) = \bigcap_{p \geq 1} H^p(D)$ is called the Arens-Hardy algebra and $H_\omega(D) \supseteq H^\infty(D)$. Then

$$d_{H_\omega}(f, g) = \sum_{p=1}^{\infty} 2^{-p} \frac{\|f - g\|_p}{1 + \|f - g\|_p}$$

is an invariant metric on $H_\omega(D)$ (see [1]).

**Theorem 2.** Suppose $B = N_+(D)$ or $H_\omega(D)$. Then $(\phi_n)$ is an $\ell_B(\phi_n)$-interpolating sequence if and only if $\inf_n \epsilon^B_n(\rho_n) > 0$.

Proof. Since $d_B$ satisfies the condition (1) and (2) in Introduction, Lemma 5 and Theorem 1 show the theorem.

§4. $\epsilon^B_n(\rho_n)$ and $\ell_B(\phi_n)$ for $B = H_\omega(D)$.

**Lemma 6.** We may assume $\alpha \neq 0$. Let $\alpha$ be a complex constant and $n$ a natural number

$$d_{H_\omega}(\alpha f_n + J, 0) = \frac{|\alpha|}{|\alpha| + |B_n(a_n)|}$$

Proof. For any $g \in J$

$$d_{H_\omega}(\alpha f_n + g, 0) = \sum_{p=1}^{\infty} 2^{-p} \frac{|\alpha|\|B_n/B_n(a_n) + g/\alpha\|_p}{1 + |\alpha|\|B_n/B_n(a_n) + g/\alpha\|_p}$$

and

$$d_{H_\omega}(\alpha f_n, 0) = \frac{|\alpha|}{|B_n(a_n)| + |\alpha|}.$$ 

Since $g = Bh$ for some $h \in H^p$, $\|B_n/B_n(a_n) + g\|_p = \|B_n(a_n)h\|_p/|B_n(a_n)|$ and so

$$\inf_{g \in J} \frac{B_n}{B_n(a_n)} + g\|_p = \frac{1}{|B_n(a_n)|}.$$ 

Hence

$$d_{H_\omega}(\alpha f_n + J, 0) \geq \frac{|\alpha|}{|B_n(a_n)| + |\alpha|}$$

because $x/(1 + x)$ is increasing on $(-1, \infty)$. This shows the lemma.
Lemma 7. For $B = H^\omega$, $\kappa_n = 1$ for any $n$.
Proof. Since $\kappa_n = \sup d_{H^\omega}(f + J_n, 0)$, by Lemma 6 $\kappa_n = 1$.

Theorem 3. For $B = H^\omega$, the following are valid.
(1) $\varepsilon_n^B(\rho_n) = 1 + |B_n(a_n)|$.
(2) $\ell_B(\phi_n) = \{(w_n) : \sum_{n=1}^{\infty} |w_n|(1 + |B_n(a_n)|)/(|w_n| + |B_n(a_n)|) < \infty\}$.
Proof. By Lemma 6, for $\alpha \in \mathbb{C}$
$$\varepsilon_n^B(\alpha) = \frac{|\alpha|(1 + |B_n(a_n)|)}{|\alpha| + |B_n(a_n)|}.$$ Since $\rho_n = \infty$ and $\varepsilon_n^B(\alpha)$ is increasing on $|\alpha| < \infty$, (1) is shown. (2) is clear.

By Theorem 2 and (1) of Theorem 3, if $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$ then $(\phi_n)$ is always an $\ell_B(\phi_n)$-interpolating sequence for $B = H^\omega$.

§5. $\varepsilon_n^B(\rho_n)$ and $\ell_B(\phi_n)$ for $B = N_+(D)$

Put $\gamma(a) = \sup\{|f(a)| : f \in N_+(D), d_{N_+}(f, 0) \leq 1\}$.

Lemma 8. Let $B = N_+(D)$, then $\rho_n = \gamma(a_n) \prod_{j=1}^{\infty} |B_j(a_j)|$ for $n = 1, 2, \ldots$.
Proof. It is proved essentially in [2, Lemma 3.1].

By the definition, $f_n = B_n/B_n(a_n)$ and $J = B_n(z)\frac{z - a_n}{1 - \bar{a}_n z}N_+$. Hence by Lemma 6,
$$\inf_{g \in J} \int_0^{2\pi} \log(1 + |\alpha||f_n(e^{i\theta}) + g(e^{i\theta})|) d\theta / 2\pi$$
$$= \inf_{g \in J} \int_0^{2\pi} \log \left(1 + \frac{|\alpha|}{|B_n(a_n)|} |B_n(e^{i\theta}) + g(e^{i\theta})| \right) d\theta / 2\pi$$
$$= \inf_{g \in N_+} \int_0^{2\pi} \log \left(1 + \frac{|\alpha|\gamma(a_n) |1 - \bar{a}_n e^{i\theta} + G(e^{i\theta})|}{\rho_n} \right) d\theta / 2\pi.$$ 

Conjecture. If $\alpha$ and $a$ are complex numbers with $\alpha \geq 0$ and $|a| < 1$, then
$$\inf_{g \in N_+} \int_0^{2\pi} \log \left(1 + \alpha \frac{|1 - \bar{a}_n e^{i\theta} + g(e^{i\theta})|}{e^{i\theta} - a} \right) d\theta / 2\pi = \log(1 + \alpha)$$
and so
$$\varepsilon_n^{N_+}(\alpha) = \log(1 + |\alpha|\gamma(a_n)/\rho_n) / \log(1 + \gamma(a_n)/\rho_n).$$
Hence if we could prove ‘Conjecture’ then \((\phi_n)\) is an \(\ell_{N+}(\phi_n)\)-interpolating sequence if and only if
\[
\inf_n \frac{\log(1 + \gamma(a_n))}{\log(1 + \gamma(a_n)/\rho_n)} > 0.
\]

We could not prove ‘Conjecture’ but we would like to know the infimum above.

It is easy to say that \(\gamma(a) \to \infty\) as \(|a| \to 1\) and \(\gamma(a) \leq \exp 2/(1-|a|)+1\) \((a \in D)\). Suppose \(\gamma_n = \gamma(a_n)\) and \(|a_n| \to 1\) as \(n \to \infty\). Then there exist \(\delta > 1\) and \(n_0\) such that \(\gamma_n^{1-\delta} \leq 2\rho_n\) \((n \geq n_0)\) if and only if \(\inf_n \log(1+\gamma_n)/\log(1+\gamma_n/\rho_n) > 0\).

In fact if \(\varepsilon = \inf_n \log(1 + \gamma_n)/\log(1 + \gamma_n/\rho_n)\) and \(\varepsilon > 0\) then \(1 + \gamma_n \geq (1 + \gamma_n/\rho_n)^{\varepsilon}\). Hence
\[
1 + \frac{1}{\gamma_n} \geq \left(\frac{1}{\gamma_n^{1/\varepsilon}} + \frac{1}{\gamma_n^{1/\varepsilon}}\right)^{\varepsilon}.
\]

and so \(\lim_{n \to \infty} \gamma_n^{1-\frac{1}{\varepsilon}}/\rho_n \leq 1\). Therefore there exist \(\delta > 1\) and \(n_0\) such that \(\gamma_n^{1-\delta} \leq 2\rho_n\) \((n \geq n_0)\). Conversely if there exist \(\delta > 1\) and \(n_0\) such that \(\gamma_n^{1-\delta} \leq 2\rho_n\) \((n \geq n_0)\) then by L’Hospital’s theorem
\[
\frac{\log(1 + \gamma_n)}{\log(1 + \frac{\gamma_n}{\rho_n})} \geq \frac{\log(1 + \gamma_n)}{\log(1 + 2\gamma_n^{\delta})} \geq \frac{1}{\delta}.
\]

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