Abstract. In this paper we present a characterization of Banach spaces possessing the Radon-Nikodym property in terms of the limit average range of additive interval functions defined on $[0,1]$ and taking values in a Frechet space.

Mathematics Subject Classification: Primary 28B05, 46B22, 58C20, Secondary 46G10, 46G05

Keywords: The limit average range, the Radon-Nikodym property, Frechet spaces

1. Introduction and Preliminaries

In the paper [7], it is shown that a Banach space $X$ has the Radon-Nikodym property (RNP) if and only if for every $X$-valued additive interval function $F : [0,1] \to X$ that has absolutely continuous McShane variational measure, the function $F$ has the limit average range at almost all $t \in [0,1]$. In this paper, we extend the last result to the Frechet spaces, Theorem 2.1.

There are also other characterizations of Banach spaces possessing the RNP in terms of additive interval functions, see [3] and [6]. A detailed study of Banach spaces possessing the RNP is presented in books [1], [4] and [5].

Throughout this paper, $X$ is a Frechet space and $\tau = \{p_k : k \in \mathbb{N}\}$ is a countable family of continuous seminorms on $X$ such that the topology of
\(X\) is generated by \(\tau\) and \(p_k \leq p_{k+1}\), for all \(k \in \mathbb{N}\). We denote by \(\tilde{X}_k\) the quotient vector space \(X/p_k^{-1}\{0\}\), by \((\tilde{X}_k, \tilde{p}_k)\) the quotient normed space and by \((\overline{X}_k, \overline{\omega}_k)\) its completion. The map \(\phi_k : X \to \tilde{X}_k\) is the canonical quotient map. Thus, \(\tilde{p}_k(\phi_k(x)) = p_k(x)\), for all \(x \in X\). For every \(k \in \mathbb{N}\), we define the map \(\tilde{g}_k : \tilde{X}_{k+1} \to \tilde{X}_k\) as follows

\[
\tilde{g}_k(w_{k+1}) = w_k \quad \text{for all} \quad w_{k+1} \in \tilde{X}_{k+1},
\]

where \(w_k = \phi_k(x)\), for some vector \(x \in w_{k+1}\). By \(\overline{\tilde{g}}_k\) the continuous linear extension of \(\tilde{g}_k\) to \(\overline{\tilde{X}}_{k+1}\) is denoted.

Let \(F : [0, 1] \to X\) be a function and let \(t \in [0, 1]\) and \(\delta > 0\). We put

\[
\Delta F(t, h) = \frac{F(t + h) - F(t)}{h}, \quad A_F(t, \delta) = \{\Delta F(t, h) : 0 < |h| < \delta\}
\]

and

\[
A_F^{(k)}(t) = \bigcap_{\delta > 0} \overline{A}_F^{(k)}(t, \delta),
\]

where \(\overline{A}_F^{(k)}(t, \delta)\) is the closure of \(A_F(t, \delta)\) with respect to the continuous semi-norm \(p_k\). The set \(A_F(t) = \bigcap_{k=1}^{+\infty} A_F^{(k)}(t)\) is said to be the average range of \(F\) at \(t\). Since \(\overline{A}_F(t, \delta) = \bigcap_{k=1}^{+\infty} \overline{A}_F^{(k)}(t, \delta)\) we have also

\[
A_F(t) = \bigcap_{\delta > 0} \overline{A}_F(t, \delta).
\]

Denote \(k\)-diam\((W) = \sup\{p_k(x - y) : x, y \in W\}\), where \(W \subset X\). We say that \(F\) has the limit average range at the point \(t\) if \(A_F(t)\) is a bounded set and for each \(\varepsilon > 0\) and each \(k \in \mathbb{N}\), there exists \(\delta^k_\varepsilon > 0\) such that

\[
k\text{-diam}(A_F(t, \delta^k_\varepsilon)) < k\text{-diam}(A_F(t)) + \varepsilon.
\]

The function \(F : [0, 1] \to X\) is said to be differentiable at \(t \in [0, 1]\), if there exists \(x \in X\) such that for each \(k \in \mathbb{N}\), we have

\[
\lim_{h \to 0} p_k(\frac{\tilde{F}(\langle t, t + h \rangle)}{|h|} - x) = 0,
\]

where \(\langle t, t + h \rangle = [\min\{t, t + h\}, \max\{t, t + h\}\]\). By \(x = F'(t)\) the derivative of \(F\) at \(t\) is denoted.

We denote by \(I\) the family of all non-degenerate closed subintervals of \([0, 1]\), by \(\lambda\) the Lebesgue measure on \([0, 1]\) and by \(\mathcal{L}\) the family of all Lebesgue measurable subsets of \([0, 1]\). If a point function \(F : [0, 1] \to X\) is given, then we denote by \(\tilde{F}\) the interval function \(\tilde{F} : \mathcal{I} \to X\) defined by \(\tilde{F}([u, v]) = F(v) - F(u)\), for all \([u, v] \in \mathcal{I}\). An interval function \(\varphi : \mathcal{I} \to X\) is said to be additive if for each two nonoverlapping intervals \(I, J \in \mathcal{I}\) with \(I \cup J \in \mathcal{I}\), we have \(\varphi(I \cup J) = \varphi(I) + \varphi(J)\). The intervals \(I, J \in \mathcal{I}\) are said to be nonoverlapping if \(\text{int}(I) \cap \text{int}(J) = \emptyset\), where \(\text{int}(I)\) denotes the interior of \(I\).
The function $F : [0,1] \to X$ is said to be **strongly absolutely continuous** (sAC) if given $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $\eta^k_\varepsilon > 0$ such that for every finite collection $\{I_1, I_2, \ldots, I_m\}$ of pairwise nonoverlapping intervals in $\mathcal{I}$, we have
\[
\sum_{i=1}^m \lambda(I_i) < \eta^k_\varepsilon \Rightarrow \sum_{i=1}^m p_k(\tilde{F}(I_i)) < \varepsilon.
\]
A pair $(I,t)$ of an interval $I \in \mathcal{I}$ and a point $t \in [0,1]$ is said to be a McShane tagged interval, $t$ is the tag of $I$. An McShane partition ($\mathcal{M}$-partition) in $[0,1]$ is a finite collection of McShane tagged intervals $(I,t)$ whose corresponding intervals are pairwise nonoverlapping. A function $\delta : A \to (0, +\infty)$ is said to be a **gauge on $A \subset [0,1]$**. We say that an $\mathcal{M}$-partition $\pi$ in $[0,1]$ is
- A-tagged if for each $(I,t) \in \pi$, we have $t \in A$,
- $\delta$-fine, if for every $(I,t) \in \pi$, we have $I \subset (t - \delta(t), t + \delta(t))$.

Given a function $F : [0,1] \to X$, $k \in \mathbb{N}$, a subset $E \subset [0,1]$ and gauge $\delta$ on $E$, we define
\[
V^\mathcal{M}_{(k,F)}(E,\delta) = \sup_{(I,t) \in \pi} \sum p_k(\tilde{F}(I)),
\]
where supremum is taken over all $E$-tagged, $\delta$-fine, $\mathcal{M}$-partition $\pi$ in $[0,1]$. Then we set
\[
V^\mathcal{M}_{(k,F)}(E) = \inf \{V^\mathcal{M}_{(k,F)}(E,\delta) : \text{ $\delta$ is a gauge on $E$} \}.
\]
The set function $V^\mathcal{M}_{(k,F)}(.)$ is said to be the **McShane variational measure** generated by $F$ with respect to seminorm $p_k$. We say that the McShane variational measure $V^\mathcal{M}_{(k,F)}$ is absolutely continuous with respect to Lebesgue measure ($V^\mathcal{M}_{(k,F)} \ll \lambda$), if $\lambda(E) = 0$ implies $V^\mathcal{M}_{(k,F)}(E) = 0$.

A countable additive vector measure $\nu : \mathcal{L} \to X$ is said to be of **bounded variation**, if for each $k \in \mathbb{N}$, we have that the countable additive vector measure $\phi_k \circ \nu$ is of bounded variation. The vector measure $\nu$ is said to be $\lambda$-**continuous**, if $\nu(E) = 0$ whenever $\lambda(E) = 0$, $E \in \mathcal{L}$. It is easy to see that $\nu$ is $\lambda$-continuous, if and only if each $\phi_k \circ \nu$ is $\lambda$-continuous.

We say that $X$ has the **Radon-Nikodym property** (the RNP) if for each countable additive vector measure $\nu : \mathcal{L} \to X$ of bounded variation and $\lambda$-continuous, there exists an integrable by seminorms function $f : [0,1] \to X$ such that $\nu(E) = \int_E f d\lambda$, for all $E \in \mathcal{L}$. We refer to [2] for information about the integrability by seminorms and the strongly integrability. These notions coincide with Bochner integrability in a Banach space.

2. A Characterization of Frechet spaces possessing the RNP

The main result is Theorem 2.1. Let us start with the following auxiliary lemma.

**Lemma 2.1.** Let $F : [0,1] \to X$ be a function. If $F$ is sAC, then there exists a unique countable additive vector measure $F_\mathcal{C} : \mathcal{L} \to X$ of bounded variation, $\lambda$-continuous and such that
\[
F(I) = F_\mathcal{C}(I) \quad \text{for every} \quad I \in \mathcal{I}.
\]
Proof. By Lemma 2.4 in [8], for each \( k \in \mathbb{N} \), there is a unique countable additive vector measure \( F^{(k)}_L : \mathcal{L} \to \overline{X}_k \) of bounded variation, \( \lambda \)-continuous and such that

\[ (\phi_k \circ F)(I) = F^{(k)}_L(I) \quad \text{for each} \quad I \in \mathcal{I}. \tag{2.1} \]

Suppose that an arbitrary \( k \in \mathbb{N} \) is given. Since \( \overline{g}_k \) is a continuous and linear map, we get that \( \overline{g}_k \circ F^{(k+1)}_L \) is a countably additive measure of bounded variation, \( \lambda \)-continuous and such that

\[ (\phi_k \circ F)(I) = (\overline{g}_k \circ F^{(k+1)}_L)(I) \quad \text{for each} \quad I \in \mathcal{I}. \]

Consequently, we obtain by uniqueness of \( F^{(k)}_L \) that \( \overline{g}_k \circ F^{(k+1)}_L = F^{(k)}_L \). It follows by Theorem II.5.4 in [10] that for every \( E \in \mathcal{L} \) there exists a unique vector \( F_L(E) \in X \) such that

\[ \phi_k(F_L(E)) = F^{(k)}_L(E) \quad \text{for all} \quad k \in \mathbb{N}. \tag{2.2} \]

Hence, the function \( F_L : \mathcal{L} \to X \) is a countable additive vector measure of bounded variation and \( \lambda \)-continuous, because each \( F^{(k)}_L \) is such a vector measure. By (2.1) and (2.2), for each \( I \in \mathcal{I} \), we have also that \( \phi_k(F_L(I)) = \phi_k(F(I)) \), for each \( k \in \mathbb{N} \). Since \( X \) is Hausdorff, the last result yields \( F_L(I) = F(I) \), for all \( I \in \mathcal{I} \). Thus, \( F_L \) is the desired vector measure and the proof is finished. \( \square \)

Now we are ready to present the main result.

**Theorem 2.1.** Let \( X \) be a Frechet space and let \( F : [0, 1] \to X \) be a function. Then the following statements are equivalent.

(i) \( X \) has the RNP,

(ii) If for each \( k \in \mathbb{N} \), we have \( V^{M}_{(k,F)} \ll \lambda \), then \( F \) has the limit average range almost everywhere on \([0, 1] \).

**Proof.** (i) \( \Rightarrow \) (ii) Assume that (i) holds and for each \( k \in \mathbb{N} \), we have \( V^{M}_{(k,F)} \ll \lambda \). Since \( V^{M}_{(k,F)} = V^{M}_{(k,F)} \), for all \( k \in \mathbb{N} \), we obtain by Lemma 3.1 in [6] that each \( \phi_k \circ F \) is sAC and therefore \( F \) is sAC. Hence, by Lemma 2.1, there exists a countable additive vector measure \( \nu : \mathcal{L} \to X \) of bounded variation, \( \lambda \)-continuous and such that

\[ \tilde{F}(I) = \nu(I) \quad \text{for every} \quad I \in \mathcal{I}, \tag{2.3} \]

and since \( X \) has the RNP, there exists an integrable by seminorm function \( f \) such that \( \nu(E) = \int_E f \, d\lambda \), for every \( E \in \mathcal{L} \). The last result together with Theorem 2.12 in [2] yields that each \( \phi_k \circ f \) is Bochner integrable and \( (\phi_k \circ \nu)(E) = (B) \int_E (\phi_k \circ f) \, d\lambda \), for each \( E \in \mathcal{L} \), and therefore by (2.3), we obtain

\[ (\phi_k \circ \tilde{F})(I) = (B) \int_I (\phi_k \circ f) \, d\lambda, \quad \text{for all} \quad I \in \mathcal{I}. \]

The last result together with Theorem II.2.9 in [5] yields that for each \( k \in \mathbb{N} \) there exists \( Z_k \subset [0, 1] \) with \( \lambda(Z_k) = 0 \) such that \( (\phi_k \circ F)'(t) \) exists for all \( t \in [0, 1] \setminus Z_k \). Hence, we obtain by Lemma 2.1 in [8] that for each \( k \in \mathbb{N} \) the function \( \phi_k \circ F \) has the limit
average range at all \( t \in [0,1] \setminus Z_k \). Consequently, we infer that \( F \) has the limit average range at all \( t \in [0,1] \setminus \bigcup_{k=1}^{\infty} Z_k \).

(ii) \( \Rightarrow \) (i) Assume that (ii) holds and let \( \nu : \mathcal{L} \to X \) be a \( \lambda \)-continuous countable additive measure of bounded variation. Let us define the function \( F : [0,1] \to X \) as follows

\[
\tilde{F}(I) = \nu(I) \quad \text{for all} \quad I \in \mathcal{I}.
\]

Fix an arbitrary \( k \in \mathbb{N} \). Since \( \phi_k \circ \nu \) is \( \lambda \)-continuous, its variation \(|\phi_k \circ \nu|\) is also \( \lambda \)-continuous, and since \(|\phi_k \circ \nu|\) is a bounded measure we obtain by Theorem 6.11 in [9] that to a given \( \varepsilon > 0 \) there exists \( \eta > 0 \) such that for every \( E \in \mathcal{L}, \) we have \( \lambda(E) < \eta \Rightarrow |\phi_k \circ \nu|(E) < \varepsilon \). Let \( D \) be a finite collection of pairwise nonoverlapping intervals in \( \mathcal{I} \) such that \( \bigcup_{I \in D} \lambda(I) < \eta \). Then, we have

\[
\sum_{I \in D} p_k((\phi_k \circ \tilde{F})(I)) = \sum_{I \in D} p_k((\phi_k \circ \nu)(I)) \leq \sum_{I \in D} |\phi_k \circ \nu|(I) = |\phi_k \circ \nu|(\bigcup_{I \in D} I) < \varepsilon.
\]

This means that \( \phi_k \circ F \) is \( sAC \) and therefore we obtain by Lemma 3.1 in [6] that \( V^{\mathcal{M}}_{\phi_k F} \ll \lambda \). Then, by Theorem 3.1 in [7], there exists a Bochner integrable function \( f_k : [0,1] \to \overline{X}_k \) such that

\[
(\phi_k \circ \tilde{F})(I) = (\phi_k \circ \nu)(I) = (B) \int_I f_k d\lambda \quad \text{for each} \quad I \in \mathcal{I}.
\]

Hence, by Theorem II.2.9 in [5], there exists \( Z_k \subset [0,1] \) with \( \lambda(Z_k) = 0 \) such that \( (\phi_k \circ \tilde{F})(t) = f_k(t) \) for all \( t \in [0,1] \setminus Z_k \). Fix an arbitrary \( t \in [0,1] \setminus \bigcup_{k=1}^{\infty} Z_k \). Since \( \overline{\gamma}_k(\phi_{k+1}(\Delta F(t,h))) = \phi_k(\Delta F(t,h)) \) we obtain

\[
\overline{\gamma}_k(f_{k+1}(t)) = f_k(t) \quad \text{for all} \quad k \in \mathbb{N}.
\]

Therefore, we get by Theorem II.5.4 in [10] that there exists an unique vector \( x_t \in X \) such that for each \( k \in \mathbb{N} \), we have \( \phi_k(x_t) = f_k(t) \). Hence, we infer that the function \( f : [0,1] \to X \) defined as follows

\[
f(t) = \begin{cases} x_t & t \in [0,1] \setminus \bigcup_{k=1}^{\infty} Z_k \\ 0 & t \in \bigcup_{k=1}^{\infty} Z_k \end{cases}
\]

is integrable by seminorms and for each \( E \in \mathcal{L}, \) we have

\[
(\phi_k \circ \nu)(E) = (B) \int_E f_k d\lambda = \phi_k(\int_E f d\lambda) \quad \text{for all} \quad k \in \mathbb{N}.
\]

Since \( X \) is Hausdorff the last result yields that

\[
\nu(E) = \int_E f d\lambda \quad \text{for each} \quad E \in \mathcal{L}.
\]

This proves that \( \nu \) has an integrable by seminorms Radon-Nikodym density \( f \) and therefore \( X \) has the RNP and the proof is finished. \( \square \)
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Received: June 25, 2013