On the Soft Sets and Algebraic Structures

Burak Kurt

Akdeniz University, Department of Mathematics
Faculty of Educations, 07058, Antalya, Turkey

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Abstract

The main purpose of this study is to introduce and investigate the basic concepts of soft set theory and soft group theory. We give some examples for the soft groups and soft subgroups.

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1 Introduction

Researchers in economics, environmental science, engineering deal with complex problems of modeling uncertain data. Classical methods are not always successful. While probability theory, fuzzy set and interval mathematical are well-known, each of these theories has its inherent difficulties as pointed out by Molodtsov [5]. Firstly Molodtsov [5] is given soft set definition and is defined some mathematical analysis on soft set.

Afterwords, P. K. Maji et al. ([3],[4]) are given soft sets examples and these are given table representation, algorithm and some sets family.

M. Irfan Ali et al. [2] and A. Sezgin et al. [6] are proved the operations of union, intersection, also De Morgan’s laws and a number of results are verified in soft set theory.

H. Aktas and N. Çağman [1] gave defining soft group and gave some example. They proved some theorem on soft subgroup.
The main purpose of this paper is to introduce of soft group theory which extends the notion of a $D_4$-Dihedral group. Also we have shown $\Gamma_u(2)$, $\Gamma_p(2)$ and $\Gamma_v(2)$ 2-order subgroup of $\Gamma(1)$ modular group, we give $\Gamma(1)$ normal soft subgroup of $\Gamma_p(2)$.

**Definition 1** $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$ such that

$$w = T(z) = \frac{az + b}{cz + d}$$

transformations in the form of called Mobius transformations.

Because of this map matrix product process, it is known that a group non-commutative. It is shown that this maps set general is $PSL(2, \mathbb{C})$.

$$\Gamma(n) := \{T: \; ad - bc = n, \; a, b, c, d \in \mathbb{Z}\}$$

equal is called $n$-order modular group. Other subgroups of $\Gamma(n)$ respectively:

$$\Gamma_0(n) := \{T: \; ad - bc = n, \; b \equiv 0 \mod n, \; a, b, c, d \in \mathbb{Z}\},$$

$$\Gamma^0(n) := \{T: \; ad - bc = n, \; c \equiv 0 \mod n, \; a, b, c, d \in \mathbb{Z}\}$$

and

$$\Gamma_0^0(n) := \{T: \; ad - bc = n, \; b \equiv c \equiv 0, \; a, b, c, d \in \mathbb{Z}\}.$$

Respectively generating of $\Gamma(1)$ modular group is infinitive order $U$ map and fourth order $V$ maps as:

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } V = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Other elements of $\Gamma(1)$

$$W = UVU = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \; P = VU = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; P^2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

In other words, respectively 2-order subgroups of $\Gamma(1)$ is $\Gamma_p(2)$, $\Gamma_u(2)$, $\Gamma_v(2)$ and $\Gamma_w(2)$ which is defined as

$$\Gamma_u(2) := \{T \in \Gamma(1): \; T \equiv I \text{ or } U \mod 2\},$$
\[ \Gamma_v(2) := \{ T \in \Gamma(1) : T \equiv I \text{ or } V \pmod{2}\} , \]
\[ \Gamma_w(2) := \{ T \in \Gamma(1) : T \equiv I \text{ or } W \pmod{2}\} \]
and
\[ \Gamma_p(2) := \{ T \in \Gamma(1) : T \equiv I, P \text{ or } P^2 \pmod{2}\} . \]
These definitions are used by Rankin [6]. More information can be obtained here.

**Definition 2** Let \( U \) initial universe, \( E \) is a set parameters set. A pair \((F,E)\) is called a soft set over \( U \) where \( F \) is a mapping given by \( F : E \to \mathcal{P}(U) \).

Furthermore soft set can be defined parameters collection of subsets of \( U \). For every \( \varepsilon \in E \), \( F(\varepsilon) \) set, \((F,E)\) can be haven \( \varepsilon \)-elements set of soft set or \( \varepsilon \)-approximate elements of soft set.

We taken soft set example which is given vintage example by Molodtsov [5].

**Example 1** Let
\[ U \text{ is the set of houses}, \]
\[ E \text{ is the set of parameters. Each parameter is a word or a sequences.} \]
\[ E = \{ \text{expensive; beautiful; wooden; cheap; in the green surroundings; modern; in good repair; in bad repair}\}. \]
In this case, shown that; \( e_1 \) expensive house, \( e_2 \) beautiful house, \( e_3 \) wooden house, \( e_4 \) cheap house, \( e_5 \) in the green surroundings house. Let
\[ F(e_1) = \{ h_2, h_4 \}, \quad F(e_2) = \{ h_1, h_3 \}, \quad F(e_3) = \{ h_3, h_4, h_5 \}, \]
\[ F(e_4) = \{ h_1, h_3, h_5 \}, \quad F(e_5) = \{ h_1 \}. \]

\((F,E)\) soft set;
\[ (F,E) = \{ \text{expensive house} = \{ h_2, h_4 \}, \text{ beautiful house} = \{ h_1, h_3 \}, \text{ wooden house} = \{ h_3, h_4, h_5 \}, \text{ cheap house} = \{ h_1, h_3, h_5 \}, \text{ in the green surroundings house} = \{ h_1 \}\}. \]

**Definition 3** Let \((F,A)\) and \((G,B)\) are two soft sets over a common universe \( U \). If it satisfies:

i. \( A \subset B \),

ii. For \( \forall \varepsilon \in A \), \( F(\varepsilon) \) and \( G(\varepsilon) \) are identical approximations
we say that \((F,A)\) is a soft subset of \((G,B)\) denoted by \((F,A) \subset (G,B)\).
Definition 4 Let \((F, A)\) and \((G, B)\) are two soft sets over a common universe \(U\). If \((F, A) \subsetneq (G, B)\) and \((G, B) \subsetneq (F, A)\), \((F, A)\) and \((G, B)\) are called soft set equal.

Example 2 Let \(A = \{e_1, e_3, e_5\} \subset E, B = \{e_1, e_2, e_3, e_5\} \subset E\). Clearly \(A \subset B\). Let \((F, A)\) and \((G, B)\) be two soft sets over the same universe \(U = \{h_1, h_2, h_3, h_4, h_5\}\) such that

\[
G(e_1) = \{h_2, h_4\}, G(e_2) = \{h_1, h_3\}, G(e_3) = \{h_3, h_4, h_5\}, G(e_5) = \{h_1\}
\]

and

\[
F(e_1) = \{h_2, h_4\}, F(e_3) = \{h_3, h_4, h_5\}, F(e_5) = \{h_1\}.
\]

Therefore \((F, A) \subsetneq (G, B)\).

Definition 5 Let \((F, A)\) and \((G, B)\) be two soft sets over a common universe \(U\). The union of \((F, A)\) and \((G, B)\) is defined to be the soft set \((H, C)\) satisfying the following conditions \(C = A \cup B, \forall e \in C\),

\[
H(e) = \begin{cases} 
F(e), & e \in A \setminus B \\
G(e), & e \in B \setminus A \\
F(e) \cup G(e), & e \in A \cap B 
\end{cases}
\]

This relation is denoted by \((F, A) \cup (G, B) = (H, C)\).

Definition 6 The intersection of two soft sets \((F, A)\), \((G, B)\) over a common universe set \(U\) is the soft set \((H, C)\), where \(C = A \cap B\) and \(\forall e \in C\), \(H(e) = F(e)\) or \(G(e)\). We write \((F, A) \cap (G, B) = (H, C)\).

Definition 7 (AND Operation On Two Soft Sets) If \((F, A)\) and \((G, B)\) are two soft sets then "\((F, A)\) AND \((G, B)\)" denoted by \((F, A) \land (G, B)\) is defined by \((F, A) \land (G, B) = (H, A \times B)\) where \(\forall (\alpha, \beta) \in A \times B\), \(H(\alpha, \beta) = F(\alpha) \cap G(\beta)\).

Definition 8 (OR Operation On Two Soft Sets) If \((F, A)\) and \((G, B)\) are two soft sets then "\((F, A)\) OR \((G, B)\)" denoted by \((F, A) \lor (G, B)\) is defined by \((F, A) \lor (G, B) = (O, A \times B)\) where \(\forall (\alpha, \beta) \in A \times B\), \(O(\alpha, \beta) = F(\alpha) \cup G(\beta)\).

2 Soft Group

Throughout this section, \(G\) is a group and \(A\) is any nonempty set. \(R\) will refer to an arbitrary binary relation between an element of \(A\) and an element of \(G\), \(A\) set-valued function \(F : A \to P(G)\) can be defined as \(F(x) = \{y \in G: (x, y) \in R, x \in A\) and \(y \in B\}\). The pair \((F, A)\) is then a soft set over \(G\). Defining a set-valued function from \(A\) to \(G\) also defines a binary relation \(R\) on \(A \times G\), given by \(R = \{(x, y) \in A \times G, y \in F(x)\}\). The triplet \((A, G, R)\) is referred to as an approximation set.
Definition 9 Let \((F, A)\) be a soft set cover \(G\). Then \((F, A)\) is said to be a soft group over \(G\) if and only if \(F(x) < G\) for all \(x \in A\).

Let this definition using the following example.

Example 3 Suppose that \(D_4\)-Dihedral group such that

\[
G = A = D_4 = \{e, (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)\}
\]

and that we define the set-valued function \(F(x) = \{y \in G : xRy \Rightarrow y = x^n, n \in N\}\).
Then the soft group \((F, A)\) is a parameterized family \(\{F(x) : x \in A\}\) of subsets which gives us a collection of subgroups of \(G\). In this case, we can view the soft group \((F, A)\) as the collection of subgroup of \(G\) given below:

\[
F(e) = \{e\}, \quad F(1234) = \{e, (13)\}, \quad F((13)(24)) = \{e, (13)\}, \\
F(1432) = \{e, (1432)\}, \quad F((12)(34)) = \{e, (12)(34)\}, \quad F((14)(23)) = \{e, (14)(23)\}, \\
F((13)) = \{e, (13)\}, \quad F((24)) = \{e, (24)\}.
\]

Theorem 1 Let \((F, A)\) and \((H, A)\) are two soft groups over \(G\). Then their intersection \((F, A) \cap (H, A)\) is a soft group over \(G\).

Proof. From Definition 6, we can write \((F, A) \cap (H, A) = (U, C)\) where \(C = A \cap A\) and \(\forall x \in C\), we have \(U(x) = F(x)\) or \(U(x) = H(x)\). \(U : A \rightarrow P(G)\) is a mapping. Therefore \((U, A)\) is a soft over \(G\). For all \(x \in A\), \(U(x) = F(x) < G\) or \(U(x) = H(x) < G\) since \((F, A)\) and \((H, A)\) are soft groups over \(G\).

Theorem 2 Let \((F, A)\) and \((H, B)\) are two soft groups over \(G\). If \(A \cap B = \emptyset\), then \((F, A) \cup (H, B)\) is a soft group over \(G\).

Proof. From Definition 5 we can write \((F, A) \cup (H, B) = (U, C)\). Since \(A \cap B = \emptyset\), it follows that either \(x \in A \setminus B\) or \(x \in B \setminus A\) for \(\forall x \in C\). If \(x \in A \setminus B\), then \(U(x) = F(x) < G\) and if \(x \in B \setminus A\) then \(U(x) = H(x) < G\). Thus, \((F, A) \cup (H, B)\) is a soft group over \(G\).

Theorem 3 Let \((F, A)\) and \((H, B)\) are two soft groups over \(G\). Then \((F, A) \wedge (H, B)\) is soft group over \(G\).

Proof. From Definition 7, we can write \((F, A) \wedge (H, B) = (U, A \times B)\). As \(F(\alpha)\) and \(H(\beta)\) are subgroups of \(G\), \(F(\alpha) \cap H(\beta)\) is a subgroup of \(G\). Therefore \(U(\alpha, \beta)\) is a subgroup of \(G\) for all \((\alpha, \beta) \in A \times B\). Hence we find that \((F, A) \wedge (H, B)\) is a soft group over \(G\).
Definition 10 Let \((F,A)\) and \((H,K)\) are two soft groups over \(G\). Then \((H,K)\) is a soft subgroup of \((F,A)\) written \((H,K) \preceq (F,A)\) if

\[
i. \quad K \subset A
\]

\[
ii. \quad H(x) \preceq F(x) \text{ for all } x \in K.
\]

Example 4 From Example 3, let \(D_4\) Dihedral group \(G\). Let \(A = \{\rho_0, \rho_1, \rho_2, \rho_3\}\) and \(K = \{\rho_0, \rho_2\}\) are two subgroup of \(G\). In here it is \(\rho_0 = e, \rho_1 = (1234), \rho_2 = (13)(24), \rho_3 = (1432)\). If we define the function 
\[F(x) = \{y \in G : (x, y) \in R, x \in A \text{ and } y \in B\},\]
\[F : A \to P(G)\] then \((H,K) \preceq (F,A)\).

Definition 11 Let \((F,A)\) be a soft group over \(G\) and \((H,B)\) be soft group of \((F,A)\). Then we say that \((H,B)\) is a normal soft subgroup of \((F,A)\) written \((H,B) \triangleleft (F,A)\), if \(H(x)\) is a normal subgroup \(F(x)\); i.e. \(H(x) \lhd F(x), \forall x \in B\).

Theorem 4 Let \((F,A)\) be a soft group over \(G\) and let \((H_i,K_i)_{i \in I}\) be a family of the normal soft subgroup of \((F,A)\). Then

\[
i. \quad \cap (H_i, K_i) \text{ is a normal soft subgroup of } (F,A),
\]

\[
ii. \quad \bigwedge_{i \in I} (H_i, K_i) \text{ is a normal soft subgroup of } (F,A),
\]

\[
iii. \quad \text{If } K_i \cap K_j \neq \emptyset \text{ for all } i, j \in I, \text{ then } \bigvee_{i \in I} (H_i, K_i) \text{ is a normal soft subgroup of } (F,A).
\]

Example 5 Let \(\Gamma(1)\) modular group such that \(G = A = \Gamma(1)\). Let \(F(x) = \{y \in G : xRy \iff y = x^n, n \in \mathbb{N}\}\) and \(B = \Gamma_p(2)\). \((F,A)\) is soft group on \(G\) and \((H,B)\) is subgroup of \((F,A)\). For \(\forall x \in B\), it is \((H,\Gamma_p(2)) \preceq (F,\Gamma(1))\) from \(H(x) \lhd F(x)\).

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References


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