Some Identities of Frobenius-Type Eulerian Polynomials Arising from Umbral Calculus

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Abstract. In this paper, we study some properties of umbral calculus related with Frobenius-type Eulerian polynomials. From our results of this paper, we can derive many interesting identities with respect to Frobenius-type Eulerian polynomials.
1. INTRODUCTION

Let $\mathbb{C}$ be the complex number field. Throughout this paper, we assume that $\lambda \in \mathbb{C}$ with $\lambda \neq 1$. The Frobenius-type Eulerian polynomials of order $r$ are given by

$$\left( \frac{1 - \lambda}{e^{(\lambda-1)t} - \lambda} \right)^r e^{xt} = \sum_{n=0}^{\infty} A_n^{(r)}(x|\lambda) t^n / n! \quad \text{ (see [1,7,8]).} \tag{1.1}$$

In the special case, $x = 0$, $A_n^{(r)}(0|\lambda) = A_n^{(r)}(\lambda)$ are called the Frobenius-type Eulerian numbers. By (1.1), we easily get

$$A_n^{(r)}(x|\lambda) = \sum_{k=0}^{\infty} \binom{n}{k} A_k^{(r)}(\lambda)x^{n-k}, \quad \text{ (see [1,3,9,11]).} \tag{1.2}$$

Let $\mathbb{P}$ be the algebra of polynomials in the single variable $x$ over $\mathbb{C}$ and $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. The action of the linear functional on a polynomial $p(x)$ is denoted by $\langle L|p(x) \rangle$. The action $\langle L|p(x) \rangle$ satisfies $\langle L + M|p(x) \rangle = \langle L|p(x) \rangle + \langle M|p(x) \rangle$ and $\langle cL|p(x) \rangle = c \langle L|p(x) \rangle$, where $c$ is a complex constant (see [10, 13, 14]).

Let $\mathcal{F}$ denote the algebra of all formal power series in the single variable $t$ over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k t^k \bigg| a_k \in \mathbb{C} \right\}. \tag{1.3}$$

For $f(t) \in \mathcal{F}$, we define a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n \rangle = a_n, \quad (n \geq 0) \quad \text{ (see [10,13,14]).} \tag{1.4}$$

By (1.3) and (1.4), we get

$$\langle t^n | x^n \rangle = n! \delta_{n,k} \quad (n, k \geq 0), \tag{1.5}$$

where $\delta_{n,k}$ is the Kronecker’s symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k \rangle}{k!} t^k$. Then, by (1.5), we easily get $\langle f_L(t)|x^n \rangle = \langle L|x^n \rangle$ and $f_L(t) = L, \quad (n \geq 0)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ is thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra (see [5, 10, 13, 14]).

The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer $k$ for which the coefficient of $t^k$ does not vanish. If $o(f(t)) = 1$, then $f(t)$ is called a delta series. If $o(f(t)) = 0$, then $f(t)$ is called an invertible series (see [5, 10, 13, 14]). Let $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\langle g(t)f(t)^k|S_n(x) \rangle = n! \delta_{n,k}$,
where \( n, k \geq 0 \). The sequence \( S_n(x) \) is called Sheffer sequence for \((g(t), f(t))\), which is denoted by \( S_n(x) \sim (g(t), f(t)) \) (see [10, 13, 14]). From (1.5), we note that \( \langle e^{yt} | p(x) \rangle = p(y) \). Let us assume that \( f(t) \in \mathcal{F} \) and \( p(x) \in \mathbb{P} \). Then, we have

\[
 f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) | x^k \rangle}{k!} t^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k | p(x) \rangle}{k!} x^k \quad \text{(see [5, 10, 13, 14]).} \tag{1.6}
\]

From (1.6), we note that

\[
 p^{(k)}(0) = \langle t^k | p(x) \rangle, \quad \langle 1 | p^{(k)}(x) \rangle = p^{(k)}(0). \tag{1.7}
\]

By (1.7), we get

\[
 t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad (k \geq 0), \quad \text{(see [5, 10, 13, 14]).} \tag{1.8}
\]

Let \( S_n(x) \sim (g(t), f(t)) \). Then we have

\[
 \frac{1}{g(f(t))} e^{yf(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \quad \text{for all } y \in \mathbb{C}, \tag{1.9}
\]

where \( \bar{f}(t) \) is the compositional inverse of \( f(t) \) (see [5, 10, 13, 14]).

The purpose of this paper is to study some properties of Frobenius-type Eulerian polynomials arising from umbral calculus. By using our results of this paper, we can obtain many interesting identities of Frobenius-type Eulerian polynomials.

2. Frobenius-type Eulerian polynomials and umbral calculus

In this section, we assume that \( r \in \mathbb{Z} \). From (1.1) and (1.9), we note that

\[
 A^{(r)}_n(x|\lambda) \sim \left( \left( \frac{e^{(\lambda-1)/t} - \lambda}{1 - \lambda} \right)^r, t \right). \tag{2.1}
\]

Let \( \mathbb{P}_n = \{ p(x) \in \mathbb{C}[x] \mid \deg p(x) \leq n \} \). Then \( \mathbb{P}_n \) is the \((n+1)\)-dimensional vector space over \( \mathbb{C} \). It is easy to show that \( \{ A^{(r)}_0(x|\lambda), A^{(r)}_1(x|\lambda), \ldots, A^{(r)}_n(x|\lambda) \} \) is a good basis for \( \mathbb{P}_n \) (see [1-17]).

For \( p(x) \in \mathbb{P}_n \), let us assume that

\[
 p(x) = \sum_{k=0}^{n} c_k A^{(r)}_k(x|\lambda), \quad (n \geq 0). \tag{2.2}
\]

Then, by (2.1) and (2.2), we get

\[
 \left\langle \left( \frac{e^{(\lambda-1)/t} - \lambda}{1 - \lambda} \right)^r t^k \middle| p(x) \right\rangle = \sum_{l=0}^{n} c_l \left\langle \left( \frac{e^{(\lambda-1)/t} - \lambda}{1 - \lambda} \right)^r t^k \middle| A^{(r)}_l(x|\lambda) \right\rangle = \sum_{l=0}^{n} c_l l! \delta_{l,k} = k! c_k. \tag{2.3}
\]
Thus, from (2.3), we have
\[
c_k = \frac{1}{k!} \left< \left( \frac{e^{t(\lambda-1)} - \lambda}{1 - \lambda} \right)^r T^k \right| p(x) \right> = \frac{1}{k!} \left< \left( \frac{e^{t(\lambda-1)} - \lambda}{1 - \lambda} \right)^r D_k^r p(x) \right>
\]
\[
= \frac{1}{k!(1 - \lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} \left< e^{j(\lambda-1)t} T^j \right| D_k^r p(x) \right>
\]
\[
= \frac{1}{k!(1 - \lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} \left< t^j D_k^r p(x + j(\lambda - 1)) \right>. 
\]

Therefore, by (2.2) and (2.4), we obtain the following theorem.

**Theorem 2.1.** For \( r \in \mathbb{Z}_+ \), \( p(x) \in \mathbb{P}_n \), let

\[
p(x) = \sum_{k=0}^{n} c_k A_k^{(r)}(x|\lambda).
\]

Then we have
\[
c_k = \frac{1}{k!(1 - \lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} D_k^r p(j(\lambda - 1)),
\]

where \( Dp(x) = \frac{dp(x)}{dx} \).

By Theorem 2.1, we get
\[
p(x) = \frac{1}{(1 - \lambda)^r} \sum_{k=0}^{n} \left\{ \sum_{j=0}^{r} \frac{1}{k!} \binom{r}{j} (-\lambda)^{r-j} D_k^r p(j(\lambda - 1)) \right\} A_k^{(r)}(x|\lambda). \quad (2.5)
\]

Let us define \( \lambda \)-difference operator \( \Delta_\lambda \) as follows:
\[
\Delta_\lambda f(x) = f(x + \lambda - 1) - \lambda f(x), \quad (2.6)
\]

and
\[
T_\lambda (f) = \frac{1}{1 - \lambda} \Delta_\lambda f(x) = \frac{1}{1 - \lambda} \left\{ f(x + \lambda - 1) - \lambda f(x) \right\}. \quad (2.7)
\]

From (2.7), we have
\[
T_\lambda \left( A_n^{(r)}(x|\lambda) \right) = \frac{1}{1 - \lambda} \left\{ A_n^{(r)}(x + \lambda - 1|\lambda) - \lambda A_n^{(r)}(x|\lambda) \right\}. \quad (2.8)
\]

By (1.1), we easily get
\[
\sum_{n=0}^{\infty} \left\{ A_n^{(r)}(x + \lambda - 1|\lambda) - \lambda A_n^{(r)}(x|\lambda) \right\} \frac{t^n}{n!} = \left( \frac{1 - \lambda}{e^{t(\lambda-1)} - \lambda} \right)^r e^{(x+\lambda-1)t} - \lambda \left( \frac{1 - \lambda}{e^{t(\lambda-1)} - \lambda} \right)^r e^{xt} \quad (2.9)
\]
\[
= (1 - \lambda) \left( \frac{1 - \lambda}{e^{t(\lambda-1)} - \lambda} \right)^{r-1} e^{xt} = (1 - \lambda) \sum_{n=0}^{\infty} A_n^{(r-1)}(x|\lambda) \frac{t^n}{n!}.
\]
Thus, by (2.9), we see that

$$T_\lambda (A_n^{(r)}(x|\lambda)) = \frac{1}{1-\lambda} \left\{ A_n^{(r)}(x+\lambda-1|\lambda) - \lambda A_n^{(r)}(x|\lambda) \right\} = A_n^{(r-1)}(x|\lambda). \quad (2.10)$$

From (2.10), we have

$$T_\lambda^{(r)}(A_n^{(r)}(x|\lambda)) = T_\lambda^{(r-1)}(A_n^{(r-1)}(x|\lambda)) = \cdots = A_n^{(0)}(x|\lambda) = x^n. \quad (2.11)$$

By (2.11), we get

$$T_\lambda^{(r)}(x^n) = T_\lambda^{(r)}(A_n^{(0)}(x|\lambda)) = A_n^{(-r)}(x|\lambda) = T_\lambda^{2r}(A_n^{(r)}(x|\lambda)). \quad (2.12)$$

For $s \in \mathbb{Z}_+$, from (2.12), we note that

$$T_\lambda^s (A_n^{(r)}(x|\lambda)) = A_n^{(r-s)}(x|\lambda). \quad (2.13)$$

On the other hand, by (2.13), we get

$$T_\lambda^s (A_n^{(r)}(x|\lambda)) = \left(\frac{e^{t(\lambda-1)} - \lambda}{1-\lambda}\right)^s (A_n^{(r)}(x|\lambda))$$

$$= \frac{1}{(1-\lambda)^s} \left( 1 - \lambda + \sum_{k=1}^{\infty} \frac{(\lambda-1)^k}{k!} k^s \right) A_n^{(r)}(x|\lambda)$$

$$= \sum_{m=0}^{s} \frac{s}{(1-\lambda)^m} \sum_{l=m}^{\infty} \frac{1}{l!} \sum_{k_1,\ldots,k_m=l, k_i \geq 1} \left( \frac{l}{k_1,\ldots,k_m} \right) D^l A_n^{(r)}(x|\lambda)$$

$$= \sum_{m=0}^{s} \frac{s}{(1-\lambda)^m} \sum_{l=m}^{\infty} \frac{1}{l!} \sum_{k_1,\ldots,k_m=l, k_i \geq 1} \left( \frac{l}{k_1,\ldots,k_m} \right) A_{n-l}^{(r)}(x|\lambda)$$

$$= \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) \sum_{m=0}^{l} \frac{s}{(1-\lambda)^m} \sum_{k_1,\ldots,k_m=l, k_i \geq 1} \left( \frac{l}{k_1,\ldots,k_m} \right) A_{n-l}^{(r)}(x|\lambda)$$

$$+ \sum_{l=\min\{s,n\}+1}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) \sum_{m=0}^{\min\{s,n\}} \frac{s}{(1-\lambda)^m} \sum_{k_1,\ldots,k_m=l, k_i \geq 1} \left( \frac{l}{k_1,\ldots,k_m} \right) A_{n-l}^{(r)}(x|\lambda). \quad (2.14)$$

Therefore, by (2.13) and (2.14), we obtain the following theorem.
Theorem 2.2. For \( r, s \in \mathbb{Z}_+ \), we have
\[
A_n^{(r-s)}(x|\lambda) = \sum_{l=0}^{\min\{r, n\}} \sum_{m=0}^{l} \sum_{k_1 + \cdots + k_m = l, k_i \geq 1} \frac{\binom{n}{m} \binom{s}{k_1} \cdots \binom{r}{k_m}}{(1-\lambda)^{m-l}} A_{n-l}^r(x|\lambda)
+ \sum_{l=\min\{s, n\}+1}^{n} \sum_{m=0}^{\min\{s, n\}} \sum_{k_1 + \cdots + k_m = l, k_i \geq 1} \frac{\binom{n}{m} \binom{s}{k_1} \cdots \binom{r}{k_m}}{(1-\lambda)^{m-l}} A_{n-l}^r(x|\lambda).
\]

Let us take \( r = s \). Then, by Theorem 2.2, we get
\[
x^n = \sum_{l=0}^{\min\{r, n\}} \sum_{m=0}^{l} \sum_{k_1 + \cdots + k_m = l, k_i \geq 1} \frac{\binom{k_1, \ldots, k_m}{l} \binom{n}{m} \binom{r}{m}}{(1-\lambda)^{m-l}} A_{n-l}^r(x|\lambda)
+ \sum_{l=\min\{r, n\}+1}^{n} \sum_{m=0}^{\min\{r, n\}} \sum_{k_1 + \cdots + k_m = l, k_i \geq 1} \frac{\binom{k_1, \ldots, k_m}{l} \binom{n}{m} \binom{r}{m}}{(1-\lambda)^{m-l}} A_{n-l}^r(x|\lambda).
\]

From (2.6), we can derive the following equation:
\[
\Delta^r_{\lambda} f(0) = \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} f((\lambda - 1)k). \tag{2.15}
\]

Let \( s = 2r \). Then, by (2.12) and Theorem 2.2, we get
\[
T^r_{\lambda}(x^n) = A_n^{(-r)}(x|\lambda) = T^{2r}_{\lambda}(A_n^r(x|\lambda))
= \sum_{l=0}^{\min\{2r, n\}} \sum_{m=0}^{l} \sum_{k_1 + \cdots + k_m = l, k_i \geq 1} \frac{\binom{k_1, \ldots, k_m}{l} \binom{n}{m} \binom{2r}{m}}{(1-\lambda)^{m-l}} A_{n-l}^r(x|\lambda)
+ \sum_{l=\min\{2r, n\}+1}^{n} \sum_{m=0}^{\min\{2r, n\}} \sum_{k_1 + \cdots + k_m = l, k_i \geq 1} \frac{\binom{k_1, \ldots, k_m}{l} \binom{n}{m} \binom{2r}{m}}{(1-\lambda)^{m-l}} A_{n-l}^r(x|\lambda). \tag{2.16}
\]

By (2.7), we easily get
\[
T^r_{\lambda}(x^n) = \frac{\Delta^r_{\lambda} x^n}{(1-\lambda)^r} = \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} (x + (\lambda - 1)j)^n. \tag{2.17}
\]

For \( n, k \geq 0 \), let us define \( \lambda \)-analogue of the Stirling number of the second kind as follows:
\[
S_2(n, k|\lambda) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-\lambda)^{k-j} j^n. \tag{2.18}
\]

From (2.18), we note that \( S_2(n, k|1) = S_2(n, k) \) where \( S_2(n, k) \) is the Stirling number of the second kind. Therefore, by (2.16), (2.17) and (2.18), we obtain the following theorem.
Theorem 2.3. For $n, k \geq 0$, we have
\[
\sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} (x + (\lambda - 1)j)^n
\]
\[
= \sum_{l=0}^{\min(2r,n)} \sum_{m=0}^{l} \sum_{k_1+\cdots+k_m=l, k_i \geq 1} \frac{(2r)^{l} \binom{l}{m} \binom{n}{l}}{(1 - \lambda)^{m-l}} A_{n-l}^{(r)}(x|\lambda)
\]
\[
+ \sum_{l=\min(2r,n)+1}^{n} \sum_{m=0}^{\min(2r,n)} \sum_{k_1+\cdots+k_m=l, k_i \geq 1} \frac{(k_1,\ldots,k_m)^{n}}{(1 - \lambda)^{m-l}} A_{n-l}^{(r)}(x|\lambda).
\]
Moreover,
\[
(\lambda - 1)^n S_2(n, r|\lambda)
\]
\[
= \frac{1}{r!} \sum_{l=0}^{\min(2r,n)} \sum_{m=0}^{l} \sum_{k_1+\cdots+k_m=l, k_i \geq 1} \frac{(k_1,\ldots,k_m)^{n}}{(1 - \lambda)^{m-l}} A_{n-l}^{(r)}(\lambda)
\]
\[
+ \frac{1}{r!} \sum_{l=\min(2r,n)+1}^{n} \sum_{m=0}^{\min(2r,n)} \sum_{k_1+\cdots+k_m=l, k_i \geq 1} \frac{(k_1,\ldots,k_m)^{n}}{(1 - \lambda)^{m-l}} A_{n-l}^{(r)}(\lambda).
\]

References

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