Fuzzy Almost Primal Ideal and
Fuzzy Strongly Almost Primal Ideal

Abdelghani Taouti 1 and Waheed Ahmad Khan 2

Department of Mathematics and Statistics
Caledonian College of Engineering, PO Box 2322
Seeb 111, Sultanate of Oman
1e-mail: ganitaouti@yahoo.com.au
2e-mail: sirwak2003@yahoo.com

Abstract. In this note we introduce Strongly almost primal element, strongly almost primal ideal and strongly almost primary ideal. Consequently we also introduce fuzzy almost primal ideal and fuzzy strongly almost primal ideal.

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1. Introduction and Preliminaries

An element $a$ is called prime to an ideal $I$, if $ab \in I$ implies $b \in I$ and an ideal $I$ is a primal ideal if the elements which are not prime to $I$ form an Ideal $P$ called the adjoint ideal of $I$ [4]. An element $a \in R$ is said to be almost prime to $I$ provided that $ra \in I - I^2$ (with $r \in R$) implies that $r \in I$. Following [1, definition 2.4], an ideal $I$ of a ring $R$ is called an almost prime if $xy \in I - I^2$ implies either $x \in I$ or $y \in I$. If $A(I)$ denote the set of all elements of $R$ that are not almost prime to $I$, $I$ is called an almost primal ideal of $R$ if the set $A(I) \cup I^2$ forms an ideal of $R$ [2]. A proper ideal of $R$ is called almost primary ideal of $R$ if whenever, $ab \in I - I^2$ then $a \in I$ or $b \in rad(I)$ [3, definition 2.1]. It is well known, an ideal $I$ of a commutative ring $R$ is said to be irreducible if
Let $\beta$ be an integral fractionary fuzzy ideal of $R$ then is strongly primary fuzzy ideal of $R$ if for any fractionary ideals $\mu$ and $\nu$ of $R$, $\mu \circ \nu \subseteq \beta$ implies that $\mu \subseteq \beta$ or $\nu \subseteq \beta$. A fuzzy ideal $\mu$ in a noetherian ring $R$ is called irreducible if $\mu \neq R$ and whenever $\mu_1 \Lambda \mu_2 = \mu$ where $\mu_1$ and $\mu_2$ are fuzzy ideals of $R$, the $\mu_1 = \mu$ or $\mu_2 = \mu$. A proper fuzzy ideal $\mu$ of a ring $R$ is said to be strongly irreducible if for each pair of fuzzy ideals $\theta$ and $\sigma$ of $R$, if $\theta \Lambda \sigma \subseteq \mu$ then either $\theta \subseteq \mu$ or $\sigma \subseteq \mu$.

Let $\mu_t = \{x \in R : \mu_t(x) \geq 1\}$, a level set, for every $t \in [0, 1]$. For a subset $W$ of $R$ let $\chi_W^{(t)}$ be the fuzzy subset of $K$ such that $\chi_W^{(t)}(x) = 1$ if $x \in W$ and $\chi_W^{(t)}(x) = t$ if $x \in K \setminus W$, where $t \in [0, 1]$. Let $R$ be an integral domain, a fuzzy $R$-submodule $\beta$ of $K$ (quotient field of $R$) is called a fractionary fuzzy ideal of $R$ if there exists $d \in R$, $d \neq 0$, such that $d_1 \circ \beta \subseteq \chi_R^{(t)}$ for some $t \in (0, 1)$. Let $\beta$ be a fractionary fuzzy ideal of $R$. Then $\beta|_R$ is a fuzzy ideal of $R$. If $\beta|_R$ is a prime(maximal) fuzzy ideal of $R$, then $\beta$ is called a prime (maximal) fractionary fuzzy ideal of $R$. If $\beta(x) = 0$ for all $x \in K \setminus R$, then $\beta$ is called an integral fractionary fuzzy ideal of $R$. Thus, if $\beta$ is a prime (maximal) integral fractionary fuzzy ideal of $R$, then $Im(\beta) = \{0, t, 1\}$ for some $t \in (0, 1)$.

In section 1, we introduce Strongly almost primal element and strongly almost primal ideal, while in section 2 we explore Fuzzy almost primal ideal and fuzzy strongly almost primal ideal in second section. we introduce fuzzy primal ideal (fuzzy almost primal), fuzzy strongly primal ideal (fuzzy almost strongly primal) and discuss its few relations among each others.

Any unexplained notation or terminology is referred to [9], [7], [12] and [14].
2. STRONGLY ALMOST PRIMAL ELEMENT AND STRONGLY ALMOST PRIMAL IDEAL

Here we introduce strongly almost prime element and strongly almost primal ideal.

We initiate with the following definition.

Definition 1. An element \( x \in K \) (ring of fractions of \( R \)) is called strongly almost prime to an ideal \( I \) of a ring \( R \) if \( xy \in I - I^2 \) implies that \( yR \subset I \). Also, an ideal \( I \) of a ring \( R \) is called strongly almost prime if \( xy \in I - I^2 \) implies either \( xR \subset I \) or \( yR \subset I \).

In case of integral domain we may define strongly almost prime element as.

Definition 2. We call an element \( x \in K \) (ring of fractions of \( R \)) a strongly almost prime to an ideal \( I \) of an integral domain \( R \) if , \( xy \in I - I^2 \) implies that \( y \in I \), for all \( x, y \in K \) (where \( K \) is a quotient field of \( R \)). Also an ideal \( I \) of an integral domain \( R \) with quotient field \( K \) is said to be strongly almost prime iff \( xy \in I - I^2 \) implies that \( x \in I \) or \( y \in I \) for all \( x, y \in K \).

Here we define an almost strongly primal ideal.

Definition 3. If \( A(I) \) represents the elements that are not strongly almost prime to an ideal \( I \) of a ring \( R \) (resp. an integral domain), \( I \) is called an strongly almost prime ideal if \( A(I) \cup I^2 \) form an ideal of \( R \).

After the above definition we will call an ideal \( A(I) \cup I^2 \), an strongly almost adjoint ideal of \( I \).

Definition 4. An ideal \( I \) of a ring (or an integral domain) \( R \) is called strongly almost primal if the elements that are not strongly almost prime to \( I \) form an ideal \( P \) called the strongly almost adjoint ideal of \( I \).

Remark 1. Let \( R \) be an integral domain, clearly \( I \) is an strongly almost primal ideal and \( P \) be strongly almost adjoint to \( I \) iff \( ab \in I, b \notin I - I^2 \) implies \( a \in I \), where \( a, b \in K \) (quotient field of \( R \)) and conversely, whenever \( a \in P \), there always exists an element \( b \) not in \( I \) such that \( ab \in I - I^2 \).

Proposition 1. In an integral domain (resp. a ring) \( R \), the product of two strongly almost prime elements to an ideal \( I \) is again a strongly almost prime element to \( I \).

Proof. Straightforward by definition1.

Remark 2. A strongly prime ideal is a strongly almost prime ideal.

Definition 5. Let \( I \) be a proper ideal of an integral domain \( R \), \( I \) is said to be strongly almost primary ideal if for \( a, b \in K \) (quotient field of \( R \)), \( ab \in I - I^2 \) implies \( a \in I \) or \( b \in \sqrt{I} \). If \( \sqrt{I} = P \) then \( I \) is called strongly almost P-primary ideal.
Remark 3. Every strongly almost prime ideal is a strongly almost primary ideal.

Proposition 2. Every strongly almost primary ideal of an integral domain $R$ is strongly almost primal ideal.

Proof. Let $I$ be a $P$-strongly almost primary ideal of an integral domain $R$. For any $a \in A(I)$, there exists $r \in K - I$ with $ra \in I - I^2$ where $r, a \in K$ (quotient field of $R$). Since $I$ is $P$-strongly almost primary ideal we have $a \in \sqrt{I} = P$ $\Rightarrow A(I) \cup I^2 \subseteq P$. Assume that $a \in P - I^2$ and suppose that $n$ is the least positive integer for which $a^n \in I - I^2$. It implies that $a$ is not an almost prime to $I$. Thus $P \subseteq A(I) \cup I^2$ and hence we have $P = A(I) \cup I^2$ i.e., $I$ is $P$-strongly almost primal ideal.

We have following implication.

Strongly prime ideal $\Rightarrow$ Strongly almost prime ideal

Strongly almost primal ideal $\Leftarrow$ Strongly almost primary ideal

From the above, no implication is reversible, however the proposition below says that when strongly almost primal ideal is a strongly almost primary ideal.

Proposition 3. A strongly almost primal ideal is a strongly primary ideal if it is a quasi-prime and also its prime radical and strongly almost adjoint (which is a strongly prime and hence a prime) prime ideal coincides.

Proof. Let us assume that $I$ is an strongly almost primal ideal and also a quasi-prime, further its prime radical is coincides with an strongly almost adjoint prime ideal $P$. We show that $I$ is a strongly almost primary ideal. Consider $ab \in I, b \notin I \Rightarrow a \in P$ (strongly almost adjoint ideal) $\Rightarrow x \in \text{rad}(I)$ (by assumption) therefore it follows that $a^n \in I$ for some $n$. Hence $I$ is an strongly almost primary ideal.

Proposition 4. A non-zero ideal $I$ of an integral domain $R$ is strongly almost primary ideal if and only if $x^{-1}I \subseteq I - I^2$ for each $x \in E(I) = \{x \in K : x^n \notin I, \text{for each } n \geq 1\}$.

Proof. If $I$ is a strongly almost primary ideal and $x \in E(I)$ then clearly $xx^{-1}I = I \Rightarrow x^{-1}I \subseteq I - I^2$. Conversely, if $yz \in I$, where $y, z \in K$ and $z \in E(I)$ then $y = zyx^{-1} \in z^{-1}I \subseteq I - I^2$.

2.1. Fuzzy almost primal ideal and fuzzy strongly almost primal ideal. In this section we introduce fuzzy (strongly) almost prime ideal, fuzzy (strongly) almost primary ideal, and fuzzy (strongly) almost primal ideal, of an integral domain. We also discuss few relations among each other.

Definition 6. A fuzzy ideal $I$ of a ring $R$ is said to be a fuzzy almost prime if for any fuzzy subsets $\mu, \nu$, $\mu \Lambda \nu \in I - I^2$ implies either $\mu \subseteq I$ or $\nu \subseteq I$. 
**Definition 7.** A fuzzy ideal $\xi$ is said to be a fuzzy almost primal ideal of commutative ring $R$, if it is non-constant and for any two fuzzy subsets $\mu$ and $\nu$ of $R$, the condition $\mu \Lambda \nu \subseteq \xi - \xi^2$ implies that $\mu \not\subseteq \xi$ and there exist a fuzzy ideal $\zeta$ such that $\nu \subseteq \zeta$. We call $\zeta$ a fuzzy almost adjoint ideal to a fuzzy primal ideal $\xi$.

**Definition 8.** A fuzzy ideal $I$ of $R$ is said to be fuzzy almost primary ideal if whenever $\mu \Lambda \nu \in I - I^2$, then $\mu \subseteq I$ or $\nu \subseteq \text{Rad}(I)$.

**Definition 9.** A prime fractionary fuzzy subset $\nu$ of an integral domain $R$ is said to be a fuzzy strongly almost prime to a fuzzy ideal $I$ of a ring $R$ if there exist fractionary fuzzy subset $\mu$ such that $\mu \circ \nu \subseteq I - I^2$ implies $\mu \subseteq I$ or $\nu \subseteq \sqrt{I}$ for all $\mu, \nu \subseteq K$ (quotient field of $R$).

**Definition 10.** Let $I$ be a proper ideal of an integral domain $R$, $I$ is said to be fuzzy strongly almost primary ideal if for $\mu, \nu \subseteq K$ (quotient field of $R$), $\mu \circ \nu \in I - I^2$ implies $\mu \subseteq I$ or $\nu \subseteq \sqrt{I}$. If $\sqrt{I} = P$ then we call $I$ is called strongly almost $P$-primary ideal.

**Proposition 5.** Fuzzy almost primary ideal of a ring $R$ is a fuzzy almost primal ideal.

**Proof.** Suppose a fuzzy ideal $\xi$ is a fuzzy almost primary ideal. Clearly, $\xi$ is non-constant and for any two fuzzy subsets $\mu, \nu$ of $R$, $\mu \Lambda \nu \subseteq \xi - \xi^2$ implies $\mu \subseteq \xi$ or $\nu \subseteq \sqrt{\xi}$. Suppose $\mu \not\subseteq \xi$ so $\nu \subseteq \sqrt{\xi} = P$ be a fuzzy almost adjoint ideal to $\xi$, such that for all $\theta, \eta, \vartheta, ...$ in ring $R$, $\theta \Lambda \nu \subseteq P$ and thus $\xi$ is a fuzzy almost primal ideal.

**Proposition 6.** Every fuzzy almost adjoint ideal to a fuzzy almost primal ideal is a fuzzy almost prime ideal.

**Proof.** Let $R$ be a commutative ring and $\zeta$ be fuzzy almost adjoint ideal to fuzzy almost primal ideal $\xi$. Following definition 7, we have $\mu \Lambda \nu \subseteq \xi - \xi^2$ implies that $\mu \not\subseteq \xi$ and there exist a fuzzy adjoint ideal $\zeta$ such that $\nu \subseteq \zeta \Rightarrow$ for all fuzzy subsets (ideals) $\theta, \eta, \vartheta, ...$ in a commutative ring $R$, not contained in $\zeta$ we have $\theta \Lambda \nu \subseteq \zeta$ (being an ideal) implies that $\nu \subseteq \zeta \Rightarrow \zeta$ is almost prime fuzzy ideal.

**Proposition 7.** In an integral domain $R$ a fuzzy strongly almost primary ideal is a fuzzy strongly almost primal ideal.

**Proof.** Let fuzzy ideal $\xi$ is a fuzzy almost strongly primary ideal, clearly $\xi$ is non-constant and for any two fractionary fuzzy ideals $\mu, \nu$ of $R$, $\mu \Lambda \nu \subseteq \xi - \xi^2$ implies $\mu \subseteq \xi$ or $\nu \subseteq \sqrt{\xi}$. Suppose $\mu \not\subseteq \xi$ so $\nu \subseteq \sqrt{\xi} = P$ be a fuzzy strongly almost adjoint ideal to $\xi$, such that for fractionary fuzzy ideals $\theta, \nu$ in a ring $R$, $\theta \Lambda \nu \subseteq P$ and thus $\xi$ is a fuzzy strongly almost primal ideal.
Abdelghani Taouti and Waheed Ahmad Khan

References


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