Domination in Hyperbolic Distributed Systems

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Abstract

In this work, we introduce and we characterize the notion of domination, for a class of hyperbolic systems. This consists to make a comparison or classification of such controlled and observed systems. We give the main properties and we consider the case of actuators sensors. Finally, as an application, we examine the case of a wave equation.

Keywords: Hyperbolic distributed systems, domination, control, observation, actuators, sensors

1 Introduction and considered systems

This work concerns the notion of domination for a general class of hyperbolic systems. It is an extension of previous works for lumped and parabolic distributed systems [1,2,3]. A more general approach is given in [5, 6, 7] for controlled and observed systems in the global, regional and asymptotic cases.
In the hyperbolic case examined in this paper, let us first consider the particular situation of a wave equation

\[
\begin{cases}
\frac{\partial^2 x}{\partial t^2}(\xi, t) = \Delta x(\xi, t) + Bu(t) & \Omega \times [0, T] \\
x(\xi, 0) = \frac{\partial x}{\partial t}(\xi, 0) = 0 & \Omega \\
x(\eta, t) = 0 & \partial \Omega \times [0, T]
\end{cases}
\]  \tag{1}

where \( \Omega \) is an open, bounded and sufficiently regular subset of \( \mathbb{R}^n \). \( B \in \mathcal{L}(U, L^2(\Omega)) \), \( u \in L^2(0, T; U) \); \( U \) is a control space, a Hilbert space; \( \Delta \) is the Laplacian operator, \( T \) is large enough. The system (1) is augmented with the following output equation

\[
y(t) = \begin{pmatrix} C_1 x(\cdot, t) \\ C_2 \frac{\partial x}{\partial t}(\cdot, t) \end{pmatrix}
\]  \tag{2}

where \( C_1 \in \mathcal{L}(L^2(\Omega), Y_1) \), \( C_2 \in \mathcal{L}(L^2(\Omega), Y_2) \), \( Y_1 \) and \( Y_2 \) are observation spaces, Hilbert spaces. Let \( A \) be the operator defined by \( A \psi = \Delta \psi \) for \( \psi \in D(A) = H^2(\Omega) \cap H^1_0(\Omega) \), and \( z = \begin{pmatrix} x \\ \frac{\partial x}{\partial t} \end{pmatrix} \in L^2(0, T; Z) \) with \( Z = H^1_0(\Omega) \times L^2(\Omega) \). The system (1) is equivalent to

\[
(\mathcal{S}) \quad \begin{cases}
\dot{z}(t) = Az(t) + Bu(t); 0 < t < T \\
z(0) = 0
\end{cases}
\]  \tag{3}

and the output equation can be written as follows

\[
y(t) = Cz(t)
\]  \tag{4}

where \( \mathcal{A} \) is the operator defined by

\[
\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}
\]  \tag{5}

with \( D(\mathcal{A}) = D(A) \times H^1_0(\Omega) \). The adjoint operator \( \mathcal{A}^* \) of \( \mathcal{A} \) is given by \( \mathcal{A}^* = -\mathcal{A} \). The operator \( \mathcal{B} \) is defined by \( \mathcal{B} = \begin{pmatrix} 0 \\ B \end{pmatrix} \), its adjoint is defined by \( \mathcal{B}^* = \begin{pmatrix} 0 & B^* \end{pmatrix} \) and \( C \in \mathcal{L}(Z, Y) \) is defined by

\[
C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \quad \text{and} \quad C^* = \begin{pmatrix} C_1^* & 0 \\ 0 & C_2^* \end{pmatrix}
\]  \tag{6}
where $Y = Y_1 \times Y_2$. The operator $A$ is linear, closed with a dense domain in the state space $Z$, and generates on $Z$ a strongly continuous semi-group (s.c.s.g) $(S(t))_{t \geq 0}$ defined by

\[
S(t) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{+\infty} \sum_{j=1}^{r_n} \langle z_1, \varphi_{nj} \rangle \Omega \cos(\sqrt{-\lambda_n} t) \\ \frac{1}{\sqrt{-\lambda_n}} \langle z_2, \varphi_{nj} \rangle \Omega \sin(\sqrt{-\lambda_n} t) \varphi_{nj} \\ \sum_{n=1}^{+\infty} \sum_{j=1}^{r_n} [\langle z_1, \varphi_{nj} \rangle \Omega \sin(\sqrt{-\lambda_n} t) \varphi_{nj} + \langle z_2, \varphi_{nj} \rangle \Omega \cos(\sqrt{-\lambda_n} t)] \end{pmatrix}
\]  

(7)

where $\langle \cdot, \cdot \rangle_\Omega$ is the inner product in $L^2(\Omega)$ and $(\varphi_{nj})_{j=1, r_n}^{n \geq 1}$ is a complete orthonormal system of eigenfunctions of $A$, associated to the eigenvalues $(\lambda_n)_{n \geq 1}$ such that $0 > \lambda_1 > \lambda_2 > \lambda_3 > \ldots$; $r_n$ is the multiplicity of $\lambda_n$ and $\sum_{n} \frac{1}{|\lambda_n|} < +\infty$. The adjoint semi-group is defined by $S^*(t) = -S(-t); \forall t \geq 0$. $Z$ is a Hilbert space with the inner product $\langle z, z' \rangle_Z = \langle (-A)^{1/2} z_1, z_1 \rangle_\Omega + \langle z_2, z'_2 \rangle_\Omega$ for $z = (z_1, z_2)$ and $z' = (z'_1, z'_2) \in Z$.

In this work, we consider more general class of hyperbolic distributed systems described by a state equation as follows

\[
(S) \begin{cases} \dot{z}(t) = Az(t) + Bu(t); 0 < t < T \\ z(0) = 0 \end{cases}
\]  

(8)

where $A$ is defined by (5), $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \in \mathcal{L}(U, Z)$, with $U = U_1 \times U_2; U_1$ and $U_2$ are two control spaces, $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in U$ and $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in Z$. The adjoint of $B$ is defined by $B^* = \begin{pmatrix} B_1^* & B_2^* \\ B_3^* & B_4^* \end{pmatrix}$. The system (8) is augmented with the output equation (4). Let $K_C$ the operator defined by

\[
K_C: u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in L^2(0, T; U) \rightarrow \int_0^T CS(T - s)Bu(s) ds \in Y
\]  

(9)
In the considered case, we have

\[
K_{C} u = \left( \begin{array}{c}
\sum_{n=1}^{\infty} \sum_{j=1}^{r_{n}} \int_{0}^{T} [\langle B_{1} u_{1}(s) + B_{2} u_{2}(s), \varphi_{n,j} \rangle_{\Omega} \cos(\sqrt{-\lambda_{n}} s) \\
\frac{1}{\sqrt{-\lambda_{n}}} \langle B_{3} u_{1}(s) + B_{4} u_{2}(s), \varphi_{n,j} \rangle_{\Omega} \sin(\sqrt{-\lambda_{n}} t)] ds C_{1} \varphi_{n,j} \\
\sum_{n=1}^{\infty} \sum_{j=1}^{r_{n}} \int_{0}^{T} [-\sqrt{-\lambda_{n}} \langle B_{1} u_{1}(s) + B_{2} u_{2}(s), \varphi_{n,j} \rangle_{\Omega} \sin(\sqrt{-\lambda_{n}} t)] ds C_{2} \varphi_{n,j} \\
\langle B_{3} u_{1}(s) + B_{4} u_{2}(s), \varphi_{n,j} \rangle_{\Omega} \cos(\sqrt{-\lambda_{n}} t)] ds C_{2} \varphi_{n,j}
\end{array} \right)
\]

Note that the system (8) is more general than the system (3) which may be obtained by considering \( B_{1} = B_{2} = B_{3} = 0 \), then \( B = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \). This paper is organized as follows: In paragraph 2, we study a possible comparison of more general controlled hyperbolic systems, with respect to an output operator \( C \). We give the main properties and characterization results. The case of sensors and actuators is also examined. Finally, we examine the case of a one dimension wave equation.

## 2 Domination for hyperbolic controlled systems

### 2.1 Problem statement and definitions

We consider the following linear distributed systems

\[
(S) \quad \left\{ \begin{array}{l}
\dot{z}(t) = Az(t) + Bu(t); 0 < t < T \\
z(0) = z_{0}
\end{array} \right.
\]  
(10)

\[
(\tilde{S}) \quad \left\{ \begin{array}{l}
\dot{\tilde{z}}(t) = \tilde{A}\tilde{z}(t) + \tilde{B}\tilde{u}(t); 0 < t < T \\
\tilde{z}(0) = \tilde{z}_{0}
\end{array} \right.
\]  
(11)

where \( A \) and \( \tilde{A} \) generate s.c.s.g \( (S(t))_{t\geq0} \) and \( (\tilde{S}(t))_{t\geq0} \) respectively; \( B \in \mathcal{L}(U, Z), \tilde{B} \in \mathcal{L}(\tilde{U}, \tilde{Z}) \), \( u \in L^{2}(0,T;U) \) and \( \tilde{u} \in L^{2}(0,T;\tilde{U}) \) with \( U = U_{1} \times U_{2} \) and \( \tilde{U} = \tilde{U}_{1} \times \tilde{U}_{2}; U_{1}, U_{2}, \tilde{U}_{1} \) and \( \tilde{U}_{2} \) are control spaces and \( Z = H^{1}_{0}(\Omega) \times L^{2}(\Omega) \).
The systems \((S)\) and \((\tilde{S})\) are augmented with the output equations

\[
y(t) = Cz(t)
\]

\[
\tilde{y}(t) = C\tilde{z}(t)
\]

where \(C \in L(Z,Y)\). The observations, are respectively given by

\[
y(T) = CS(T)z_0 + K_Cu \quad \text{and} \quad \tilde{y}(T) = C\tilde{S}(T)z_0 + \tilde{K}_C\tilde{u}
\]

\(K_C\) and \(\tilde{K}_C\) are the operators defined as follows

\[
K_C : \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in L^2(0,T;U) \quad \longrightarrow \quad \int_0^T CS(T-s)B_u(s)ds \in Y
\]

\[
\tilde{K}_C : \quad \tilde{u} = \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} \in L^2(0,T;\tilde{U}) \quad \longrightarrow \quad \int_0^T C\tilde{S}(T-s)\tilde{B}\tilde{u}(s)ds \in Y
\]

**Definition 1** We say that

i. \((S)\) dominates \((\tilde{S})\) (or the pair \((A,B)\) dominates \((\tilde{A},\tilde{B})\)) exactly on \([0,T]\), with respect to the operator \(C\), if \(\text{Im}(\tilde{K}_C) \subset \text{Im}(K_C)\).

ii. \((S)\) dominates \((\tilde{S})\) (or the pair \((A,B)\) dominates \((\tilde{A},\tilde{B})\)) weakly on \([0,T]\), with respect to the operator \(C\), if \(\text{Im}(\tilde{K}_C) \subset \overline{\text{Im}(K_C)}\).

In this case, we note respectively \((A,B) \leq_C (\tilde{A},\tilde{B})\) and \((A,B) \preccurlyeq_C (\tilde{A},\tilde{B})\).

Let us give the following properties and remarks which are similar to those obtained in the case of parabolic systems.

1. In the case where \(C\) is the identity operator \(I\), we say that \((S)\) dominates \((\tilde{S})\) (or the pair \((A,B)\) dominates \((\tilde{A},\tilde{B})\)) exactly (respectively weakly). Then we note \((A,B) \leq C (\tilde{A},\tilde{B})\) and \((A,B) \preccurlyeq C (\tilde{A},\tilde{B})\).

2. Obviously, the exact domination with respect to an output operator \(C\), implies the weak one with respect to \(C\). The converse is not true.

3. If the system \((S)\) is controllable exactly (respectively weakly), or equivalently \(\text{Im}(\mathcal{H}) = Z\) (respectively \(\overline{\text{Im}(\mathcal{H})} = Z\)), where \(\mathcal{H}\) is defined by

\[
\mathcal{H} \equiv \mathcal{H}(B) : \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in L^2(0,T;U) \quad \longrightarrow \quad \int_0^T S(T-s)Bu(s)ds \in Y
\]

then \((S)\) dominates exactly (respectively weakly) any system \((\tilde{S})\), with respect to any output operator \(C\).
4. In the case where $A = \tilde{A}$, if $(S)$ dominates $(\tilde{S})$ exactly (respectively weakly), we say simply that $B$ dominates $\tilde{B}$ exactly (respectively weakly). Then, we note $B \leq C (\tilde{B})$ (respectively $B \lessdot C (\tilde{B})$).

Hence, one can consider a single system with two inputs as follows

\[
(S) \begin{cases}
\dot{z}(t) = Az(t) + Bu(t) + \tilde{B}\tilde{u}(t); & 0 < t < T \\
z(0) = z_0 \in \mathbb{Z}
\end{cases}
\]  

augmented with an output equation

\[
(E) \quad y(t) = Cz(t)
\]  

In this case, the domination of control operators $B$ and $\tilde{B}$, with respect to the observation operator $C$, is similar. The definitions and results remain practically the same.

4. The exact or weak domination of systems (or operators) is a transitive and reflexive relation, but it is not antisymmetric. Thus, for example in the case where $A = \tilde{A}$, for any operator $B \neq 0$ and $\alpha \neq 0$, we have $\text{Im}(CH(B)) = \text{Im}(CH(\alpha B))$, even if $B \neq \alpha B$ for $\alpha \neq 1$.

6. Concerning the relationship with the notion of remediability [4, 8], we consider without loss of generality, a class of hyperbolic distributed systems described by a state equation as follows

\[
\begin{cases}
\dot{z}(t) = Az(t) + Bu(t) + d(t); & 0 < t < T \\
z(0) = z_0
\end{cases}
\]  

where $d \in L^2(0, T; \mathbb{Z})$ is a known or unknown disturbance. The system (19) is augmented with the following output equation

\[
y(t) = Cz(t)
\]  

The state $z$ of the system at time $T$ is given by

\[
z(T) = S(T)z_0 + Hu + Rd \quad \text{where} \quad Rd = \int_0^T S(T - s)d(s)ds
\]

If the system (19), augmented with (20), is exactly (respectively weakly) remediablc on $[0, T]$, or equivalently $\text{Im}(CR) \subset \text{Im}(CH)$ (respectively $\overline{\text{Im}(CR)} \subset \overline{\text{Im}(CH)}$), then $B$ dominates any operator $\tilde{B}$ exactly (respectively weakly) with respect to the operator $C$.

We give hereafter characterization results concerning the exact and weak domination.
2.2 Characterizations

The following result gives a characterization of the exact domination with respect to the output operator $C$. We assume that $Z = H^1_0(\Omega) \times L^2(\Omega)$ and that $A$ and $\tilde{A}$ generate respectively the s.c.s.g. defined by

$$
S(t) \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) = \left( \begin{array}{c}
\sum_{n=1}^{+\infty} \sum_{j=1}^{r_n} [\langle z_1, \varphi_{n_j} \rangle \Omega \cos(\sqrt{-\lambda_n t})] \\
+ \frac{1}{\sqrt{-\lambda_n}}\langle z_2, \varphi_{n_j} \rangle \Omega \sin(\sqrt{-\lambda_n t})] \varphi_{n_j}
\end{array} \right)
$$

(21)

$$
\tilde{S}(t) \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) = \left( \begin{array}{c}
\sum_{n=1}^{+\infty} \sum_{j=1}^{s_n} [\langle z_1, \psi_{n_j} \rangle \Omega \cos(\sqrt{-\gamma_n t})] \\
+ \frac{1}{\sqrt{-\gamma_n}}\langle z_2, \psi_{n_j} \rangle \Omega \sin(\sqrt{-\gamma_n t})] \psi_{n_j}
\end{array} \right)
$$

(22)

where $\{\varphi_{n_j}, j = 1, \ldots, r_n; n \geq 1\}$ (respectively $\{\psi_{n_j}, j = 1, \ldots, s_n; n \geq 1\}$) is a complete orthonormal system of eigenfunctions of $A$ (respectively $\tilde{A}$), associated to the real eigenvalues $(\lambda_n)_{n \geq 1}$ such that $0 > \lambda_1 > \lambda_2 > \lambda_3 > \ldots$, where $r_n$ is the multiplicity of $\lambda_n$ (respectively $(\gamma_n)_{n \geq 1}$, with $0 > \gamma_1 > \gamma_2 > \gamma_3 > \ldots$ where $s_n$ is the multiplicity of $\gamma_n$) with $\sum_n \frac{1}{|\lambda_n|} < +\infty$ and $\sum_n \frac{1}{|\gamma_n|} < +\infty$.

We give hereafter, a characterization of the exact domination.

**Proposition 2** The following properties are equivalent

i. The system $(S)$ dominates $(\tilde{S})$ exactly on $[0, T]$, with respect to $C$. 


ii. For any \( \tilde{u} \in L^2(0, T; \tilde{U}) \), there exists \( u \in L^2(0, T; U) \) such that

\[
K_C u + \tilde{K}_C \tilde{u} = 0
\]  

(23)

iii. There exists \( \alpha > 0 \) such that for any \( \theta \equiv (\theta_1, \theta_2) \in \mathcal{Y}' \), we have

\[
\left\| \sum_{n \geq 1} \sum_{j=1}^{r_n} \left[ (C^*_1 \theta_1, \psi_{n_j}) \Omega \cos(\sqrt{-\gamma_n} t) - \frac{1}{\sqrt{-\gamma_n}} (C^*_2 \theta_2, \psi_{n_j}) \Omega \sin(\sqrt{-\gamma_n} t) \right] \tilde{B}_3^* \psi_{n_j} \right\|_{L^2(0, T)} \\
+ (\sqrt{-\gamma_n} (C^*_1 \theta_1, \psi_{n_j}) \Omega \sin(\sqrt{-\gamma_n} t) + (C^*_2 \theta_2, \psi_{n_j}) \Omega \cos(\sqrt{-\gamma_n} t) \tilde{B}_3^* \psi_{n_j}) \\
+ \sum_{n \geq 1} \sum_{j=1}^{r_n} \left[ (C^*_1 \theta_1, \psi_{n_j}) \Omega \cos(\sqrt{-\gamma_n} t) - \frac{1}{\sqrt{-\gamma_n}} (C^*_2 \theta_2, \psi_{n_j}) \Omega \sin(\sqrt{-\gamma_n} t) \right] \tilde{B}_4^* \psi_{n_j} \\
+ (\sqrt{-\gamma_n} (C^*_1 \theta_1, \psi_{n_j}) \Omega \sin(\sqrt{-\gamma_n} t) + (C^*_2 \theta_2, \psi_{n_j}) \Omega \cos(\sqrt{-\gamma_n} t) \tilde{B}_4^* \psi_{n_j}) \right\|_{L^2(0, T; U)} \\
\leq \alpha
\]

Let us note that
- if \( \mathcal{A} = \tilde{\mathcal{A}} \), then \( \gamma_n = \lambda_n \); \( r_n = s_n \) and \( \psi_{n_j} = \varphi_{n_j} \) for \( n \geq 1 \) and \( 1 \leq j \leq r_n \).
- in the case where \( \tilde{\mathcal{A}} = a \mathcal{A} + bI \), with a convenient choice of the reals \( a \) and \( b \), we have: \( \gamma_n = a \lambda_n + b \); \( r_n = s_n \) and \( \psi_{n_j} = \varphi_{n_j} \) for \( n \geq 1 \) and \( 1 \leq j \leq r_n \).

Concerning the weak domination, we have the following proposition.

**Proposition 3** The system \((\mathcal{S})\) dominates \((\tilde{\mathcal{S}})\) weakly on \([0, T]\), with respect to \(C\), if and only if

\[
\ker[B^* S^*(T - .) C^*] \subset \ker[\tilde{B}^* \tilde{S}^*(T - .) C^*]
\]  

(24)

In the next section, we examine the case of a finite number of actuators, and then the case where the observation is given by sensors.
2.3 Case of actuators and sensors

This section is focused on the notions of actuators and sensors [8, 10, 11]. We consider without loss of generality, the case where

\[ \mathcal{B} = \begin{pmatrix} 0 \\ B \end{pmatrix} \quad \text{and} \quad \widetilde{\mathcal{B}} = \begin{pmatrix} 0 \\ \widetilde{B} \end{pmatrix} \]

2.3.1 Case of actuators

If the system (S) is excited by \( p \) zone actuators \((\Omega_i, g_i)_{1 \leq i \leq p}\), we have \( U = \mathbb{R}^p \) and \( Bu(t) = \sum_{i=1}^{p} g_i u_i(t) \), i.e. \( Bu(t) = (0, \sum_{i=1}^{p} g_i u_i(t))^{tr} \), where \( u = (u_1, \ldots, u_p)^{tr} \) and \( g_i \in L^2(\Omega_i) \); \( \Omega_i = supp(g_i) \subset \Omega \). We have

\[ B^*z = (\langle g_1, z_2 \rangle, \ldots, \langle g_p, z_2 \rangle)^{tr}, \quad \mathcal{B}^* = (0, B^*) \]

By the same, if \( \tilde{(S)} \) is excited by \( q \) zone actuators \((\tilde{\Omega}_i, \tilde{g}_i)_{1 \leq i \leq q}\), we have \( \tilde{U} = \mathbb{R}^q \) and \( \tilde{B}\tilde{u}(t) = \sum_{i=1}^{q} \tilde{g}_i \tilde{u}_i(t) \) with \( \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_q)^{tr} \) and \( \tilde{B}^*z = (\langle \tilde{g}_1, z_2 \rangle, \ldots, \langle \tilde{g}_q, z_2 \rangle)^{tr} \) and \( \tilde{B}^* = (0, \tilde{B}^*) \).

We give hereafter a preliminary characterization result. It concerns a necessary and sufficient condition for weak domination in the particular case where the operator \( C = I \). But let us first give the following lemma deriving from Fourier transform.

**Lemma 4** If \( \sum_{n} a_n \cos(\sqrt{-\lambda_n} t) = 0 \) and \( \sum_{n} b_n \sin(\sqrt{-\lambda_n} t) = 0, \forall t > 0, \)

with \( \sum_{n} a_n \) and \( \sum_{n} b_n \) absolutely convergent, then \( a_n = b_n = 0, \forall n \geq 1. \)

**Proposition 5** The system \((S)\) dominates \((\tilde{S})\) weakly on any time interval \([0, T]\), if and only if

\[ \bigcap_{n \geq 1} Ker[M_n f_n] \subset \bigcap_{n \geq 1} Ker[\widetilde{M}_n \tilde{f}_n] \]

(25)

where \( M_n = (\langle g_i, \varphi_{nj} \rangle)_{1 \leq i \leq p; 1 \leq j \leq r_n} \); \( \widetilde{M}_n = (\langle \tilde{g}_i, \psi_{nj} \rangle)_{1 \leq i \leq q; 1 \leq j \leq s_n} \)

are the controllability matrices corresponding to the actuators \((\Omega_i, g_i)_{1 \leq i \leq p}\) and \((\tilde{\Omega}_i, \tilde{g}_i)_{1 \leq i \leq q}\) respectively, and

\[ f_n(w) = \begin{pmatrix} \langle w, \varphi_{n1} \rangle \\ \vdots \\ \langle w, \varphi_{nr_n} \rangle \end{pmatrix} \quad ; \quad \tilde{f}_n(w) = \begin{pmatrix} \langle w, \psi_{n1} \rangle \\ \vdots \\ \langle w, \psi_{ns_n} \rangle \end{pmatrix} \]
Proof. We assume that $\text{Ker}(B^*S^*(.)) \subset \text{Ker}(\tilde{B}^*\tilde{S}^*(.))$. We have

$$B^*S^*(t)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \sum_{n \geq 1, j=1}^{r_n} \left( \sqrt{-\lambda_n} \langle z_1, \varphi_{nj} \rangle \langle g_i, \varphi_{nj} \rangle_{1 \leq i \leq p} \sin(\sqrt{-\lambda_n} t) \right) \\ + \sum_{j=1}^{r_n} \langle z_2, \varphi_{nj} \rangle \langle g_i, \varphi_{nj} \rangle_{1 \leq i \leq p} \cos(\sqrt{-\lambda_n} t) \right) \end{pmatrix}_{1 \leq i \leq p}$$

therefore, $(z_1, z_2)^{tr} \in \text{Ker}(B^*S^*(.))$ if and only if, for $1 \leq i \leq p$ and $t > 0$

$$\forall n \geq 1 \begin{cases} \sum_{j=1}^{r_n} \sqrt{-\lambda_n} \langle z_1, \varphi_{nj} \rangle \langle g_i, \varphi_{nj} \rangle_{1 \leq i \leq p} \sin(\sqrt{-\lambda_n} t) = 0 \\ \sum_{j=1}^{r_n} \langle z_2, \varphi_{nj} \rangle \langle g_i, \varphi_{nj} \rangle \cos(\sqrt{-\lambda_n} t) = 0 \end{cases}$$

Using Lemma 4, we have for $n \geq 1$ and $1 \leq i \leq p$

$$\sum_{j=1}^{r_n} \langle z_1, \varphi_{nj} \rangle \langle g_i, \varphi_{nj} \rangle = 0 \quad \text{and} \quad \sum_{j=1}^{r_n} \langle z_2, \varphi_{nj} \rangle \langle g_i, \varphi_{nj} \rangle = 0$$

i.e. for any $n \geq 1$, we have $f_n(z_1) \in \text{Ker}(M_n)$ and $f_n(z_2) \in \text{Ker}(M_n)$. Since $\text{Ker}(B^*S^*(.)) \subset \text{Ker}(\tilde{B}^*\tilde{S}^*(.))$, we deduce by the same that $\forall n \geq 1$

$$\tilde{f}_n(z_1) \in \text{Ker}(\tilde{M}_n) \quad \text{and} \quad \tilde{f}_n(z_2) \in \text{Ker}(\tilde{M}_n)$$

Conversely, if $\bigcap_{n \geq 1} \text{Ker}[M_nf_n] \subset \bigcap_{n \geq 1} \text{Ker}[\tilde{M}_nf_n]$, obviously $\text{Ker}(B^*S^*(.)) \subset \text{Ker}(\tilde{B}^*\tilde{S}^*(.))$.

In the case where

$$r_n = s_n \quad \text{and} \quad \psi_{nj} = \varphi_{nj} \quad \text{for} \quad 1 \leq j \leq r_n \quad \text{and} \quad n \geq 1 \quad (26)$$

in particular if $\tilde{A} = A$, then $f_n = \tilde{f}_n$ for $n \geq 1$. As mentioned before, (26) may be verified even if $\tilde{A} \neq A$. In such a situation, we have the following more practical result.

**Corollary 6** The system $(S)$ dominates $(\tilde{S})$ weakly on any interval $[0, T]$, if and only if $\text{Ker}[M_n f_n] \subset \text{Ker}[\tilde{M}_n f_n]$; $\forall n \geq 1$, or equivalently

$$\text{Ker}[M_n] \subset \text{Ker}[\tilde{M}_n] ; \quad \forall n \geq 1$$
In the general case, we give hereafter characterizations of exact and weak domination for hyperbolic systems with respect to an output operator $C$.

**Proposition 7** The system $(S)$ dominates $(\tilde{S})$ exactly on $[0, T]$, with respect to $C$, if and only if, there exists $\alpha > 0$ such that for any $\theta \equiv (\theta_1, \theta_2)^{tr} \in \mathcal{Y}$, we have

$$
\left\| \sum_{n \geq 1} \left[ \sum_{j=1}^{s_n} \sqrt{-\gamma_n} \langle C^*_1 \theta_1, \psi_{nj} \rangle \Omega \langle g_i, \psi_{nj} \rangle \right] \sin(\sqrt{-\gamma_n} t) \right\|_{L^2(0, T; \mathbb{R}^q)} \\
+ \left\| \sum_{j=1}^{r_n} \langle C^*_2 \theta_2, \psi_{nj} \rangle \Omega \langle g_i, \psi_{nj} \rangle \cos(\sqrt{-\gamma_n} t) \right\|_{L^2(0, T; \mathbb{R}^p)} 
\leq \alpha
$$

**Proof.** Derives from proposition 2. \hfill \qed

**Proposition 8** The system $(S)$ dominates $(\tilde{S})$ weakly with respect to $C$, on any interval $[0, T]$, if and only if,

$$
\bigcap_{n \geq 1} \text{Ker}[M_n P_{n,k}] \subset \bigcap_{n \geq 1} \text{Ker}[\tilde{M}_n \tilde{P}_{n,k}], \quad k = 1, 2
$$

where $P_{n,k}(\theta) = \left( \langle C^*_k \theta_k, \varphi_{nj} \rangle \right)_{j=1, \ldots, r_n}$ and $\tilde{P}_{n,k}(\theta) = \left( \langle C^*_k \theta_k, \psi_{nj} \rangle \right)_{j=1, \ldots, s_n}$

**Proof.** We assume that $\text{Ker}(B^* S^* (.) C^*) \subset \text{Ker}(\tilde{B}^* \tilde{S}^* (.) C^*)$, therefore, $\theta \equiv (\theta_1, \theta_2)^{tr} \in \text{Ker}(B^* S^* (.) C^*)$ if and only if,

$$
\sum_{n \geq 1} \left[ \sum_{j=1}^{r_n} \sqrt{-\lambda_n} \langle C^*_1 \theta_1, \varphi_{nj} \rangle \Omega \langle g_i, \varphi_{nj} \rangle \right] \sin(\sqrt{-\lambda_n} t) \\
+ \left( \sum_{j=1}^{s_n} \langle C^*_2 \theta_2, \varphi_{nj} \rangle \Omega \langle g_i, \varphi_{nj} \rangle \right) \cos(\sqrt{-\lambda_n} t) = 0, \quad \text{for } i \in \{1, \ldots, p\}; \quad \forall \ t > 0
$$
Using lemma 4, we have for \( n \geq 1 \) and \( i \in \{1, ..., p\} \)

\[
\sum_{j=1}^{r_n} \langle C^*_1 \theta_1, \varphi_{n_j} \rangle \Omega(g_i, \varphi_{n_j}) = 0 \quad \text{and} \quad \sum_{j=1}^{r_n} \langle C^*_2 \theta_2, \varphi_{n_j} \rangle \Omega(g_i, \varphi_{n_j}) = 0
\]

i.e. \( \forall n \geq 1 \), we have \( P_{n,k}(\theta) = (\langle C^*_k \theta_k, \varphi_{n_j} \rangle)_{j=1,...,r_n} \in \text{Ker}(M_n), \ k = 1, 2. \)

Using the same method for \( \theta \equiv (\theta_1, \theta_2)^{tr} \in \text{Ker}(\bar{B}^* \bar{S}^* C^*) \), we deduce that

\[
\forall n \geq 1, \ k = 1, 2; \ \text{we have} \ (\langle C^*_k \theta_k, \varphi_{n_j} \rangle)_{j=1,...,r_n} \in \text{Ker}(M_n) ] \implies \text{that}
\]

\[
\forall n \geq 1, k = 1, 2; \ \text{we have} \ (\langle C^*_k \theta_k, \psi_{n_j} \rangle)_{j=1,...,s_n} \in \text{Ker}(\tilde{M}_n].
\]

We then have the result.

We examine hereafter the case of sensors.

### 2.3.2 Case of sensors

Now, if the output of the system is given by \( q_1 \) and \( q_2 \) sensors \( (D_i, h_i)_{1 \leq i \leq q_1} \) and \( (D'_i, k_i)_{1 \leq i \leq q_2} \), we have \( C^*_1 = \sum_{l=1}^{q_1} \theta^l_1 h_l \) and \( C^*_2 = \sum_{l=1}^{q_2} \theta^l_2 k_l \) for \( \theta_1 = (\theta^l_1)_{l=1,...,q_1} \in \mathbb{R}^{q_1} \) and \( \theta_2 = (\theta^l_2)_{l=1,...,q_2} \in \mathbb{R}^{q_2} \). In this case, the exact and weak domination are equivalent. The following result gives a necessary and sufficient condition for domination of sensors.

**Proposition 9** (\( S \)) dominates (\( \tilde{S} \)) on any interval \([0, T]\), with respect to the sensors \((D_i, h_i)_{1 \leq i \leq q_1}\) and \((D'_i, k_i)_{1 \leq i \leq q_2}\), if and only if

\[
\bigcap_{n \geq 1} \text{Ker}(M_n G^\text{tr}_{n,k}) \subset \bigcap_{n \geq 1} \text{Ker}(\tilde{M}_n \tilde{G}^\text{tr}_{n,k}); \ k = 1, 2 \tag{27}
\]

where, for \( k = 1, 2; G_{n,k} \) and \( \tilde{G}_{n,k} \) are the observability matrices defined by

\[
G_{n,1} = (\langle h_{i,j}, \varphi_{n_j} \rangle)_{\{i=1,...,q_1\};\{j=1,...,r_n\}}
\]

\[
G_{n,2} = (\langle k_{i,j}, \varphi_{n_j} \rangle)_{\{i=1,...,q_2\};\{j=1,...,r_n\}}
\]

\[
\tilde{G}_{n,1} = (\langle h_{i,j}, \psi_{n_j} \rangle)_{\{i=1,...,q_1\};\{j=1,...,s_n\}}
\]

\[
\tilde{G}_{n,2} = (\langle k_{i,j}, \psi_{n_j} \rangle)_{\{i=1,...,q_2\};\{j=1,...,s_n\}}
\]
Proof. Let us first note that

\[
\mathcal{B}^* S^*(t) C^* \theta = \left( \sum_{n \geq 1} \left( \sum_{j=1}^{r_n} \sum_{l=1}^{q_1} \theta_1^l \sqrt{-\lambda_n} \langle h_l, \varphi_{n_j} \rangle \Omega \langle g_i, \varphi_{n_j} \rangle \sin(\sqrt{-\lambda_n} t) \right) + \left( \sum_{j=1}^{r_n} \sum_{l=1}^{q_2} \theta_2^l \langle k_l, \varphi_{n_j} \rangle \Omega \langle g_i, \varphi_{n_j} \rangle \cos(\sqrt{-\lambda_n} t) \right) \right)_{1 \leq i \leq p}
\]

and

\[
\widetilde{\mathcal{B}}^* \tilde{S}^*(t) C^* \theta = \left( \sum_{n \geq 1} \left( \sum_{j=1}^{s_n} \sum_{l=1}^{q_1} \theta_1^l \sqrt{-\gamma_n} \langle h_l, \psi_{n_j} \rangle \Omega \langle \tilde{g}_i, \psi_{n_j} \rangle \sin(\sqrt{-\gamma_n} t) \right) + \left( \sum_{j=1}^{s_n} \sum_{l=1}^{q_2} \theta_2^l \langle k_l, \psi_{n_j} \rangle \Omega \langle \tilde{g}_i, \psi_{n_j} \rangle \cos(\sqrt{-\gamma_n} t) \right) \right)_{1 \leq i \leq p}
\]

Then, \((S)\) dominates \(\tilde{(S)}\) on any interval \([0, T]\), with respect to \(C\), if and only if \(\ker(\mathcal{B}^* S^*(.), C^*) \subset \ker(\widetilde{\mathcal{B}}^* \tilde{S}^*(.), C^*)\), then \(\theta \equiv (\theta_1, \theta_2)^{tr} \in \ker(\mathcal{B}^* S^*(.), C^*)\) if and only if,

\[
\sum_{n \geq 1} \left( \sum_{j=1}^{r_n} \sum_{l=1}^{q_1} \theta_1^l \sqrt{-\lambda_n} \langle h_l, \varphi_{n_j} \rangle \Omega \langle g_i, \varphi_{n_j} \rangle \sin(\sqrt{-\lambda_n} t) \right) = 0; \forall t > 0
\]

and

\[
\sum_{n \geq 1} \left( \sum_{j=1}^{r_n} \sum_{l=1}^{q_2} \theta_2^l \langle k_l, \varphi_{n_j} \rangle \Omega \langle g_i, \varphi_{n_j} \rangle \cos(\sqrt{-\lambda_n} t) \right) = 0; \forall t > 0
\]

we deduce that, \(\forall n \geq 1\)

\[
\sum_{j=1}^{r_n} \sum_{l=1}^{q_1} \theta_1^l \langle h_l, \varphi_{n_j} \rangle \Omega \langle g_i, \varphi_{n_j} \rangle 1 \leq i \leq p = 0 \quad \text{and} \quad \sum_{j=1}^{r_n} \sum_{l=1}^{q_2} \theta_2^l \langle k_l, \varphi_{n_j} \rangle \Omega \langle g_i, \varphi_{n_j} \rangle 1 \leq i \leq p = 0
\]

Therefore, \(\theta \in \ker(\mathcal{B}^* S^*(.), C^*)\), implies that

\[
\theta_1 \in \bigcap_{n \geq 1} \ker(M_n G_{n,1}^{tr}) \quad \text{and} \quad \theta_2 \in \bigcap_{n \geq 1} \ker(M_n G_{n,2}^{tr})
\]

and hence (using analogous developments for \(\tilde{\theta} \in \ker(\widetilde{\mathcal{B}}^* \tilde{S}^*(.), C^*)\))

\[
\bigcap_{n \geq 1} \ker(M_n G_{n,1}^{tr}) \subset \bigcap_{n \geq 1} \ker(\tilde{M}_n \tilde{G}_{n,1}^{tr}) \quad \text{and} \quad \bigcap_{n \geq 1} \ker(M_n G_{n,2}^{tr}) \subset \bigcap_{n \geq 1} \ker(\tilde{M}_n \tilde{G}_{n,2}^{tr})
\]

Consequently, we have the result. \(\square\)
3 Application to the wave equation

To illustrate the previous results, we consider without loss of generality, a one dimension hyperbolic system described by the following wave equation,

\[
\begin{align*}
\mathcal{S}_g : \quad & \frac{\partial^2 x}{\partial t^2}(\xi, t) = \frac{\partial^2 x}{\partial \xi^2}(\xi, t) + g(\xi) u(t) \quad ]0, 1[\times]0, T[ \\
x(\xi, 0) = & \quad \frac{\partial x}{\partial t}(\xi, 0) = 0 \quad ]0, 1[ \\
x(0, t) = & \quad x(1, t) = 0 \quad ]0, T[
\end{align*}
\]

with \(\Omega = ]0, 1[\). Then the following operator \(A\) defined by \(A f = \frac{\partial^2 f}{\partial \xi^2}\), with

\[
D(A) = \{ f \in L^2(0, 1)/ f, \frac{\partial f}{\partial \xi} \text{ absolutely continuous}, \frac{\partial^2 f}{\partial \xi^2} \in L^2(0, 1) \text{ and } f(0) = f(1) = 0 \}
\]

\((-A)\) is self-adjoint, positive, bounded and invertible. Moreover, the eigenvalues of \(A\) are \(\lambda_n = -n^2\pi, \ n \geq 1\). The corresponding eigenfunctions are defined by \(\varphi_n(\xi) = \sqrt{2} \sin(n\pi \xi)\). In this case, the operator \(\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\)

generates a s.c.s.g. defined by

\[
S(t) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{+\infty} [\langle z_1, \varphi_n \rangle \cos(n\pi t) + \frac{1}{n\pi} \langle z_2, \varphi_n \rangle \sin(n\pi t)] \varphi_n \\ \sum_{n=1}^{+\infty} [-n\pi \langle z_1, \varphi_n \rangle \sin(n\pi t) + \langle z_2, \varphi_n \rangle \cos(n\pi t)] \varphi_n \end{pmatrix}
\]

In this case, it is sufficient to consider an interval \([0, T]\) with \(T \geq 2\). \(\mathcal{S}_g\) is excited by a zone actuator \((\Omega, g)\), then we have \(U = \mathbb{R}\) and \(Bu(t) = g(.)u(t)\).

The considered system is augmented with following output equation

\[
y(t) = Cz(t) \text{ where } z(t) = \begin{pmatrix} x(., t) \\ \frac{\partial x(., t)}{\partial t} \end{pmatrix}
\]

The system \(\mathcal{S}_g\) denotes the system \(\mathcal{S}_g\) by replacing \(g\) by \(\tilde{g}\). The characterizations of exact and weak domination remain true on any interval \([0, T]\) with \(T \geq 2\). They may be deduced easily from the previous results with \(\tilde{A} = A\), \(r_n = s_n = 1\) and \(\varphi_n = \psi_n\).

In the particular case where \(C = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\), we have the following result.
Proposition 10
(S_g) dominates (S_{\tilde{g}}) weakly on \([0, T]\), if and only if, for any \(n \geq 1\), we have

\[\langle g, \varphi_n \rangle = 0 \implies \langle \tilde{g}, \varphi_n \rangle = 0\]

Now, if (S_g) and (S_{\tilde{g}}) are augmented with the output equation

\[y(t) = \left( \frac{\langle h, x(., t)\rangle}{\langle k, \frac{\partial x}{\partial t}(., t)\rangle} \right)\]

using proposition 7, we deduce the following characterization.

Proposition 11 (\(\Omega, g\)) dominates (\(\Omega, \tilde{g}\)) with respect to the considered sensors, if and only if

\[\forall n \geq 1 \left\{ \begin{array}{c}
\langle g, \varphi_n \rangle \langle h, \varphi_n \rangle = 0 \\
\langle g, \varphi_n \rangle \langle k, \varphi_n \rangle = 0
\end{array} \right\} \implies \forall n \geq 1 \left\{ \begin{array}{c}
\langle \tilde{g}, \varphi_n \rangle \langle h, \varphi_n \rangle = 0 \\
\langle \tilde{g}, \varphi_n \rangle \langle k, \varphi_n \rangle = 0
\end{array} \right\} \quad (28)\]

These results can be also applied to a two-dimension space domain (a rectangle for example) and extended to more general situations and systems.

References


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