A Note on the Associated Bitopological Space \((X, \tau_1^*, \tau_2^*)\)

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Abstract. The present paper deals with zero – dimensional spaces, pairwise completely normal spaces, continuity, pairwise \(S_1\) spaces , pairwise disconnected spaces, pairwise \(T_\frac{1}{2}\), pairwise \(T_\frac{3}{2}\), pairwise \(R_\ell\).

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1. Introduction

Ivan Reilly discussed pairwise \(T_0, T_1, T_2, T_3\) and \(T_4\) – spaces in [3]. In the year 1964, J. C. Kelly [4] introduced bitopological spaces. Some aspects of bitopological spaces are given in [1] by K.Chandrasekhar Rao and in [5].This paper is a continuation of investigations done by previous authors and we study some properties of \((X, \tau_1^*, \tau_2^*)\).

We require the following known definitions:
Definitions ( [1], [2] and [4] )

* A bitopological space \((X, \tau_1, \tau_2)\) is said to be zero – dimensional if 
\((X, \tau_1)\) has a base, whose elements are \(\tau_1\)-open sets and \(\tau_2\)-closed sets, and 
\((X, \tau_2)\) has a base, whose elements are \(\tau_2\)-open sets and \(\tau_1\)-closed sets.

* A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise completely
normal provided that whenever \(A\) and \(B\) are subsets of \(X\) such that 
\([\tau_1 - \text{cl}(A)] \cap B = \emptyset\) and \([\tau_2 - \text{cl}(B)] \cap A = \emptyset\) there exists a \(\tau_2\)
open set \(U\) and a \(\tau_1\)-open set \(V\) such that \(A \subset U, B \subset V, U \cap V = \emptyset\).

* Let \((X, \tau_1, \tau_2)\) be bitopological space. If \(X = A \cup B\) where \(A\)
and \(B\) satisfy \((A \cap \tau_1 - \text{cl}(B)) \cup (\tau_2 - \text{cl}(A) \cap B) = \emptyset\), then \(X\) is
called pairwise disconnected space. Whenever \(A\) and \(B\) satisfy above
equation then \(A\) and \(B\) are said to be pairwise separated sets.

* \((X, \tau_1, \tau_2)\) is pairwise \(S_1\), if every singleton \(\{x\}\) is either \(\tau_1\)-open
set (or) \(\tau_2\)-closed set.

* Let \((X, \tau_1, \tau_2)\) be any given bitopological space. Let \(\mathcal{B}_1\) be the
family of all \((1, 2)\) – regularly open subsets of \(X\) and \(\mathcal{B}_2\) be the family
of all \((2, 1)\) – regularly open subsets of \(X\). Note that \(A\) is \((1, 2)\) -
regularly open if \(A = \tau_1 - \text{int}(\tau_2 - \text{cl} A)\). Since the intersection of two
\((i, j)\) – regularly open subsets of \(X\) is again an \((i, j)\) – regularly open
set for i, j = 1, 2 and \(i \neq j\), \(\mathcal{B}_1\) and \(\mathcal{B}_2\) both generate topologies for \(X\),
say, \(\tau_1^*\) and \(\tau_2^*\) respectively.

Thus with every given bitopological space \((X, \tau_1, \tau_2)\), there is
associated another bitopological space \((X, \tau_1^*, \tau_2^*)\).

Obviously, \(\tau_1^* \subset \tau_1\) and \(\tau_2^* \subset \tau_2\).

* If \((X, \tau_1)\) is \(T_1\) or \((X, \tau_2)\) is \(T_1\), then \((X, \tau_1, \tau_2)\) is called a \(T_{1\frac{1}{2}}\)
-space.

* If \((X, \tau_1, \tau_2)\) is pairwise \(T_{1\frac{1}{2}}\) if for \(x \neq y\) in \(X\) then there exists a \(\tau_1\)-open set \(U\)
and a \(\tau_2\)-open set \(V\) such that \(U \cap V = \emptyset\) and \(x \in U, y \in V\) (or) \(x \in V, y \in U\).
* (X, τ₁, τ₂) is said to be pairwise Urysohn if for x ≠ y in X, there exists a τ₁-open nbd U of x and there exists a τ₂-open nbd V of y such that \((τ₂-cl\ U) \cap (τ₁-cl\ V) = \phi\).

* If (X, τ₁, τ₂) is pairwise \(R_1\) then there exists a τ₁-open nbd U of x and there exists a τ₂-open nbd V of y such that U \(\cap\ V = \phi\) and τ₂-cl\{x\} ≠ τ₁-cl\{y\}.

2. Known Results

**Theorem 2.1** (X, τ₁, τ₂) is pairwise \(T_2\) \(\Rightarrow\) (X, τ⁺₁, τ⁺₂) is pairwise \(T_1\) (see [5]).

**Theorem 2.2** If (X, τ₁, τ₂) is pairwise Urysohn, then (X, τ⁺₁, τ⁺₂) is pairwise \(T_2\) (see [5]).

3. Main Results

In this section we deal with (X, τ⁺₁, τ⁺₂)

**Theorem 3.1** If (X, τ₁, τ₂) is zero – dimensional, then (X, τ⁺₁, τ⁺₂) is also zero – dimensional.

**Proof:**

Let p \(\in\) X and let \(U^*\) be any \(τ⁺₁\)-open neighbourhood of p. But \(τ⁺₁ \subset τ₁\). Hence \(U^*\) is a \(τ₁\)-open neighbourhood of p. But (X, τ₁, τ₂) is zero – dimensional. Hence there exists a set \(A\) which is \(τ₁\)-open and \(τ₂\)-closed with \(p \in A \subset U^*\). Choose \(A^*\) such that \(p \in A^*\), \(A^*\) is \(τ⁺₁\)-open and \(τ⁺₂\)-closed with \(A^* \subset A\).

We have \(p \in A^* \subset U^*\) \ ...(1)

Similarly, given a \(τ⁺₂\)-open neighbourhood \(V^*\) of p, choose a set \(B^*\), which is \(τ⁺₂\)-open and \(τ⁺₁\)-closed with \(p \in B^* \subset V^*\) \ ...(2)
From (1) and (2) it follows that \((X, \tau_1^*, \tau_2^*)\) is zero – dimensional.

**Theorem 3.2** If \((X, \tau_1^*, \tau_2^*)\) is zero – dimensional then \((X, \tau_1, \tau_2)\) is also zero – dimensional.

**Proof:**

Let \(p \in X\) and let \(U\) be any \(\tau_1\) - open neighbourhood of \(p\). But \(\tau_1^* \subseteq \tau_1\). Choose a \(\tau_1^*\) - open neighbourhood \(U^*\) of \(p\) with \(U^* \subseteq U\).

But \((X, \tau_1^*, \tau_2^*)\) is zero – dimensional. Hence there exists a set \(A^*\) which is \(\tau_1^*\) - open and \(\tau_2^*\) - closed with \(p \in A^* \subseteq U^*\). Take \(A = A^*\).

Then \(A\) is \(\tau_1\) - open and \(\tau_2\) - closed with \(p \in A \subseteq U\) ...(1)

Similarly, given a \(\tau_2\) - open neighbourhood \(V\) of \(p\), choose a set \(B\) which \(\tau_2\) - open and \(\tau_1\) - closed such that \(p \in B \subseteq V\) ...(2)

From (1) and (2) it follows that \((X, \tau_1, \tau_2)\) is also zero – dimensional.

**Theorem 3.3** If \((X, \tau_1, \tau_2)\) is pairwise completely normal then \((X, \tau_1^*, \tau_2^*)\) is pairwise completely normal.

**Proof :**

Let \((X, \tau_1, \tau_2)\) be pairwise completely normal.

We have \(\tau_1^* \subseteq \tau_1\) and \(\tau_2^* \subseteq \tau_2\). Suppose that \(A, B\) be pairwise separated sets in \((X, \tau_1^*, \tau_2^*)\).

\[ [\tau_1^* - \text{cl}(A)] \cap B \cup [\tau_2^* - \text{cl}(B)] \cap A = \phi. \]

\[ [\tau_1 - \text{cl}(A)] \cap B \cup [\tau_2 - \text{cl}(B)] \cap A = \phi. \]

\[ A, B\ are\ pairwise\ separated\ sets\ in\ \((X, \tau_1, \tau_2)\). \]

\[ \Rightarrow \] There exists a \(\tau_2\) - open set \(U\) with \(A \subseteq U\) and there exists a \(\tau_1\) - open set \(V\) with \(B \subseteq V\) with \(U \cap V = \phi\).

Take \(U^* = U\) whenever \(U \in \tau_2^*\) and \(V^* = V\) whenever \(V \in \tau_1^*\).

Then \(U^* \cap V^* = \phi\) with \(U^* \in \tau_2^*\) and \(V^* \in \tau_1^*\).

Also \(A \subseteq U^*\) and \(B \subseteq V^*\).

Hence \((X, \tau_1^*, \tau_2^*)\) is pairwise completely normal.
Theorem 3.4 Let \( i : (X, \tau_1, \tau_2) \rightarrow (X, \tau_1^*, \tau_2^*) \) be the identity function. Then \( i \) is \((\tau_1, \tau_1^*)\) continuous and \((\tau_2, \tau_2^*)\) continuous.

**Proof:** Let \( i : (X, \tau_1, \tau_2) \rightarrow (X, \tau_1^*, \tau_2^*) \) be the identity function. We have \( \tau_1^* \subset \tau_1 \). Take \( G \in \tau_1^* \). Then \( G \in \tau_1 \) \((1)\). But \( i \) is the identity map. Hence \( i^{-1}(G) = G \in \tau_1 \) by \((1)\).

Thus \( i^{-1}(G) \in \tau_1 \) if \( G \in \tau_1^* \).

Hence \( i \) is \( (\tau_1, \tau_1^*) \) continuous.

Similarly, \( i \) is also \( (\tau_2, \tau_2^*) \) continuous.

Theorem 3.5 If \( (X, \tau_1^*, \tau_2^*) \) is pairwise S\(1\) then \( (X, \tau_1, \tau_2) \) is pairwise S\(1\).

**Proof:**

Let \( (X, \tau_1^*, \tau_2^*) \) be pairwise S\(1\).

\( \Rightarrow \) Either \( \{x\} \) is \( \tau_1^* \)-open or \( \tau_2^* \)-closed.

But \( \tau_1^* \subset \tau_1 \) and \( \tau_2^* \subset \tau_2 \).

\( \Rightarrow \) Either \( \{x\} \) is \( \tau_1 \)-open or \( \tau_2 \)-closed.

Thus \( (X, \tau_1, \tau_2) \) is pairwise S\(1\).

Theorem 3.6 If \( (X, \tau_1^*, \tau_2^*) \) is pairwise disconnected then \( (X, \tau_1, \tau_2) \) is pairwise disconnected.

**Proof:** Suppose that \( (X, \tau_1^*, \tau_2^*) \) is pairwise disconnected.

\( \Rightarrow \) \( X = A \cup B \) where \( A \in \tau_1^* \), \( B \in \tau_2^* \), \( A \neq \emptyset \), \( B \neq \emptyset \) with \( A \cap B = \emptyset \).

\( \Rightarrow \) \( X = A \cup B \) where \( A \in \tau_1 \), \( B \in \tau_2 \), \( A \neq \emptyset \), \( B \neq \emptyset \) with \( A \cap B = \emptyset \), because \( \tau_1^* \subset \tau_1 \) and \( \tau_2^* \subset \tau_2 \).

\( \Rightarrow \) \( (X, \tau_1, \tau_2) \) is pairwise disconnected.

Theorem 3.7 If \( (X, \tau_1, \tau_2) \) is pairwise \( T_{1/2} \) then \( (X, \tau_1^*, \tau_2^*) \) is pairwise \( T_{1/2} \).
Proof:
Suppose that \((X, \tau_1, \tau_2)\) is pairwise \(T_{\frac{1}{2}}\).
\[\Rightarrow (X, \tau_1)\) is \(T_{\frac{1}{2}}\).
\[\Rightarrow x \neq y \text{ in } X \text{ gives a } \tau_1\text{-open set } U \text{ such that } x \in U \text{ and } y \notin U.
Take \(U^* = X - \tau_1\text{-cl } V\).
We shall show that \(x \in U^*\).
Also \(x \in U \Rightarrow y \notin V\).
But \(V \subset \tau_1\text{-cl } V\).
Hence \(x \notin \tau_1\text{-cl } V\).
\[\Rightarrow x \in X - (\tau_1\text{-cl } V).
\[\Rightarrow x \in U^*, \text{ but } y \notin U^*.
Next we shall show that \(U^* \in \tau_1^*\).
Let \(U \in \tau_1 \Rightarrow U = \tau_1\text{-int } U\). But \(U \subset \tau_2\text{-cl } U\).
\[\Rightarrow \tau_1\text{-int } U \subset \tau_1\text{-int } (\tau_2\text{-cl } U) \in \tau_1^*.
But by definition \(U^* = \tau_1\text{-int } (\tau_2\text{-cl } U)\).
Hence \(U^* \in \tau_1^*\). Thus \((X, \tau_1^*)\) is \(T_1\).
Similarly, \(V^* = X - (\tau_2\text{-cl } V)\). Then \(y \in V^*, x \notin V^*\) and \(V^* \in \tau_2^*\).
Thus \((X, \tau_1^*)\) is \(T_1\) or \((X, \tau_2^*)\) is \(T_1\).
Hence \((X, \tau_1^*, \tau_2^*)\) is pairwise \(T_{\frac{1}{2}}\).

Theorem 3.8 If \((X, \tau_1, \tau_2)\) is pairwise \(T_{\frac{1}{2}}\), then \((X, \tau_1^*, \tau_2^*)\) is pairwise \(T_{\frac{1}{2}}\).

Proof:
Suppose that \((X, \tau_1, \tau_2)\) is pairwise \(T_{\frac{1}{2}}\).
\[\Rightarrow \text{ For } x \neq y \text{ in } X, \text{ there exists a } \tau_1\text{-open set } U \text{ and there exists a } \tau_2\text{-open set } V \text{ such that } U \cap V = \phi, \text{ } x \in U \text{ and } y \in V \text{ (or) } x \in V \text{ and } y \in U.
Take \(U^* = X - (\tau_1\text{-cl } V), V^* = X - (\tau_2\text{-cl } V)\).
Then \(U^* \in \tau_1^*, V^* \in \tau_2^*, U^* \cap V^* = \phi \text{ with } x \in U^* \text{ and } y \in V^* \text{ (or) } x \in V^* \text{ and } y \in U^*.
Thus \((X, \tau_1^*, \tau_2^*)\) is pairwise \(T_{\frac{1}{2}}\).
Theorem 3.9 If \((X, \tau_1, \tau_2)\) is pairwise \(R_i\) then \((X, \tau_1^*, \tau_2^*)\) is pairwise \(R_i\).

Proof:

Suppose that \((X, \tau_1, \tau_2)\) is pairwise \(R_i\).

Let \(x \neq y\) in \(X\). Since \((X, \tau_1, \tau_2)\) is pairwise \(R_i\), there exists a \(\tau_1\)-open nbd \(U\) of \(x\) and there exists a \(\tau_2\)-open nbd \(V\) of \(y\) such that \(U \cap V = \emptyset\) and \(\tau_2\)-cl\{\(x\}\} \neq \tau_1\)-cl\{\(y\}\}.

Take \(U^* = \tau_1\)-int \((\tau_2\)-cl \(U)\)
\(V^* = \tau_2\)-int \((\tau_1\)-cl \(V)\).

If \(x \notin U^*\), then \(x \notin \tau_1\)-int \((\tau_2\)-cl \(U)\).

\(\Rightarrow\) \(x \notin \tau_2\)-cl \(U \Rightarrow x \notin U\), a contradiction. Hence \(x \in U^*\).

Similarly, \(y \in V^*\).

Also \(U^* \in \tau_1^*, V^* \in \tau_2^*\) and \(U^* \cap V^* = \emptyset\).

We have \(\tau_2^*\)-cl\{\(x\}\} \subset \tau_2\)-cl\{\(x\}\}.
\(\tau_1^*\)-cl\{\(x\}\} \subset \tau_1\)-cl\{\(x\}\}.

Also \(\tau_2\)-cl\{\(x\}\} \neq \tau_1\)-cl\{\(x\}\}.

Hence \(\tau_2^*\)-cl\{\(x\}\} \neq \tau_1^*\)-cl\{\(x\}\}. Thus \((X, \tau_1^*, \tau_2^*)\) is pairwise \(R_i\).

References


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