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A Generalized Nadler-Type Theorem in Partial Metric Spaces

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Abstract

Nadler's Fixed Point Theorem for Multi-valued functions (see [6]) was generalized by H. Aydi et al in their recent paper [1]. We use the approach of Gordji et. al. [3] to further generalize this result for a larger class of multifunctions (with a more general contraction condition) on complete partial metric spaces. The result yields the main theorem in [1] as a special case.

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1 Introduction

The concept of partial metric spaces as a generalization of metric spaces was introduced in 1994 by S.G. Matthews [5], in his treatment of denotational semantics of dataflow networks. In his definition of "distance," self-distance need not be 0. Some applications of partial metrics to problems in theoretical Computer Science, including the use of fixed point theorems to determine program output from partially defined information, are cited in [4] and references cited there. On the other hand, Banach's well-known fixed point theorem for contraction mappings on complete metric spaces has been applied to various problems, including that of establishing the existence of solutions to differential and integral equations, and has been generalized in many ways. Of note is the generalization of Nadler [6] to multifunctions on metric spaces, with the multi-valued function satisfying a contraction condition. Nadler's result has been generalized as well, see [3] and [2], among others. Recently, an analogue for contractive mutifunctions on partial metric spaces was established by H. Aydi et al. In this paper we will establish a further generalization of the result of Aydi et al.

2 Preliminary Notes

We will be dealing with multifunctions on a partial metric space throughout this paper.

Definition 2.1. *Let X be a nonempty set. A function $p : X \times X \to \Re^+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold: (P1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$; (P2) $p(x,x) \leq p(x,y)$; (P3) $p(x,y) = p(y,x)$; (P4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$. The pair (X,p) is then called a partial metric space.*

If $p(x,y) = 0$, then $x = y$. But the converse does not always hold. A standard example of a partial metric space is the pair $(\Re^+, p)$, where $p : \Re^+ \times \Re^+ \to \Re^+$ is defined as $p(x,y) = max\{x,y\}$.

A partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$. Observe from [5], p. 187 that a sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$, with respect to τ_p , if and only if $p(x,x) = lim_{n\to\infty} p(x,x_n)$. If p is a partial metric on X, then the function $p^s : X \times X \to \Re^+$ given by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ defines a metric on X. Further, a sequence x_n converges in (X,p^s) to a point $x \in X$ if and only if $lim_{n,m\to\infty} p(x_n,x_m) = lim_{n\to\infty} p(x_n,x) = p(x,x)$.

From Matthews [5], we have the following definition and lemma:

Definition 2.2. *Let (X,p) be a partial metric space. (a) A sequence x_n in X is said to be a Cauchy sequence if $lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite. (b) (X,p) is said to be complete if every Cauchy sequence x_n in X converges with respect to τ_p to a point $x \in X$ such that $lim_{n\to\infty} p(x,x_n) = p(x,x)$. In this case, we say that the partial metric p is complete.*

Lemma 2.3. *Let (X,p) be a partial metric space. Then: (a) A sequence x_n in X is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s). (b) A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete.*

Towards generalizing Nadler's Theorem to multifunctions in partial metric spaces, a partial Hausdorff metric is defined and its properties investigated by Aydi et al in [1]: In all cases, assume (X,p) is a partial metric space.

Let $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of the partial metric space (X,p). Closedness is in the setting of (X,τ_p) where τ_p is the topology induced by p, and boundedness is given as follows: A is a bounded subset in (X,p) if there exist $x_0 \in X$ and $M > 0$ such that for all $a \in A$, we have $a \in B_p(x_0,M)$, that is, $p(x_0,a) < p(x_0,x_0) + M$.

For $A, B \in CB^p(X)$ and $x \in X$, define $p(x, A) = inf\ \{p(x, a), a \in A\}$, $\delta_p(A, B) = sup\{p(a, B) : a \in A\}$ and $\delta_p(B, A) = sup\{p(b, A) : b \in B\}$. It is noted that $p(x, A) = 0$ only if $p^s(x, A) = 0$ where $p^s(x, A) = inf\{p^s(x, a), a \in A\}$.

Remark 2.4. *Let (X, p) be a partial metric space and A any nonempty set in (X, p), then $a \in \overline{A}$ if and only if $p(a, A) = p(a, a)$, where $\overline{A}$ denotes the closure of A with respect to the partial metric p.*

The following properties of the mapping $\delta_p : CB^p(X) \times CB^p(X) \to [0,\infty)$ are established in [1]:

Proposition 2.5. *For any $A, B, C \in CB^p(X)$, we have the following: (i) $\delta_p(A, A) = sup\{p(a, a) : a \in A\}$; (ii) $\delta_p(A, A) \leq \delta_p(A, B)$; (iii) $\delta_p(A, B) = 0$ implies that $A \subseteq B$; (iv) $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - inf_{c\in C} p(c, c)$.*

Let (X, p) be a partial metric space. For $A, B \in CB^p(X)$, define $H_p(A, B) = max\{\delta_p(A, B), \delta_p(B, A)\}$.

Proposition 2.6. *For all $A, B, C \in CB^p(X)$, we have (h1) $H_p(A, A) \leq H_p(A, B)$; (h2) $H_p(A, B) = H_p(B, A)$; (h3) $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - inf_{c\in C} p(c, c)$.*

Corollary 2.7. *Let (X, p) be a partial metric space. For $A, B \in CB^p(X)$ the following holds $H_p(A, B) = 0$ implies that $A = B$.*

Remark 2.8. *The converse of the above Corollary is not true in general.*

The mapping $H_p : CB^p(X) \times CB^p(X) \to [0, +\infty)$ is then called a partial Hausdorff metric induced by p.

Remark 2.9. *Any Hausdorff metric is a partial Hausdorff metric. The converse is not true.*

3 Main Results

The following is needed in the proof of the main result.

Lemma 3.1. *Let (X, p) be a partial metric space, $A, B \in CB^p(X)$ and $r > 0$. For any $a \in A$, there exists $b = b(a) \in B$ such that $p(a, b) \leq H_p(A, B) + r$.*

Proof. If $A = B$, $H_p(A, B) = H_p(A, A) = \delta_p(A, A) = \sup\{p(a, A)|a \in A\}$. Let $a \in A$. Then $p(a, a) \leq \sup_{x\in A} p(x, x) = H_p(A, B)$, hence the inequality $p(a, b) \leq H_p(A, B) + r$ is satisfied (trivially). Suppose $A \neq B$. Let $a \in A$ be such that $p(a, b) > H_p(A, B) + r$ for all $b \in B$. Then, $inf\{p(a, y)|y \in B\} \geq H_p(A, B) + r$. That is, $p(a, B) \geq H_p(A, B) + r$. Note that $H_p(A, B) \geq \delta_p(A, B) = \sup_{x\in A} p(x, B) \geq p(a, B) \geq H_p(A, B) + r$. But with $A \neq B$, $H_p(A, B) \neq 0$, and the above inequality yields a contradiction. Hence no such a exists. □

Theorem 3.2. *Let (X,p) be a complete partial metric space and let $T : X \to CB^p(X)$ be a multi-valued mapping such that for all $x, y \in X$,*

$$H_p(Tx,Ty) \leq \alpha p(x,y) + \beta[p(x,Tx) + p(y,Ty)] + \gamma[p(x,Ty) + p(y,Tx)] \quad (1)$$

$\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point in X.

Proof. Let $r := \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)}$. First assume $r > 0$. Let $x_0 \in X$ and $x_1 \in X$ with $x_1 \in Tx_0$. According to the preceding Lemma, there exists $x_2 \in Tx_1$ such that $p(x_1,x_2) \leq H_p(Tx_0,Tx_1) + r$. Further, there exists $x_3 \in Tx_2$ such that $p(x_2,x_3) \leq H_p(Tx_1,Tx_2) + r^2$. Continuing in this fashion, there exists $x_{n+1} \in Tx_n$, such that $p(x_n,x_{n+1}) \leq H_p(Tx_{n-1},Tx_n) + r^n$. So, invoking inequality 1 we have

$$\begin{aligned} p(x_n,x_{n+1}) \leq H_p(Tx_{n-1},Tx_n) + r^n \leq \alpha p(x_{n-1},x_n) + \beta[p(x_{n-1},Tx_{n-1}) + p(x_n,Tx_n)] \\ + \gamma[p(x_{n-1},Tx_n) + p(x_n,Tx_{n-1})] + r^n. \quad (2) \end{aligned}$$

But note that $p(x_{n-1},Tx_{n-1}) \leq p(x_{n-1},x_n)$ and $p(x_n,Tx_n) \leq p(x_n,x_{n+1})$ since by definition, $p(x,A) = inf\{p(x,a), a \in A\}$, and in general, $x_{n+1} \in Tx_n$. Note that $p(x_n,Tx_{n-1}) \leq p(x_n,x_n)$, and also by the triangle inequality on partial metric spaces,

$$p(x_{n-1},Tx_n) \leq p(x_{n-1},x_{n+1}) \leq p(x_{n-1},x_n) + p(x_n,x_{n+1}) - p(x_n,x_n).$$

We then arrive at the inequality

$$p(x_n,x_{n+1}) \leq \alpha p(x_{n-1},x_n) + \beta[p(x_{n-1},x_n) + p(x_n,x_{n+1})] + \gamma[p(x_{n-1},x_n) + p(x_n,x_{n+1})] + r^n.$$

Thus, $(1-\beta-\gamma)p(x_n,x_{n+1}) \leq (\alpha+\beta+\gamma)p(x_{n-1},x_n) + r^n$, or

$$p(x_n,x_{n+1}) \leq rp(x_{n-1},x_n) + \frac{r^n}{[1-(\beta+\gamma)]}$$

for all natural numbers n. From here, we get

$$p(x_n,x_{n+1}) \leq r^n p(x_0,x_1) + \frac{nr^n}{[1-(\beta+\gamma)]}$$

for all n. Since $r < 1$, $\sum_{n=1}^{\infty} p(x_n,x_{n+1}) < \infty$ and with $p^s(x_n,x_{n+m}) \leq 2p(x_n,x_{n+m}) \to 0$ as $n \to \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence in the metric space (X,p^s). Since (X,p) is complete, (X,p^s) is complete by Lemma 1.6 in [1], and the sequence $\{x_n\}$ converges (in the complete metric space (X,p^s) to a point x^* in X, i.e., $\lim_{n\to\infty} p^s(x_n,x^*) = 0$. Again, from (1.1)

of [1], $p(x^*, x^*) = \lim_{n\to\infty} p(x_n, x^*) = \lim_{n\to\infty} p(x_n, x_n) = 0$. To show that x^* is a fixed point of T, note that

$$\begin{aligned} p(x^*, Tx^*) &\le p(x^*, x_{n+1}) + p(x_{n+1}, Tx^*) \\ &\le p(x^*, x_{n+1}) + H_p(Tx_n, Tx^*) \\ &\le p(x^*, x_{n+1}) + \alpha p(x_n, x^*) + \beta[p(x_n, Tx_n) + p(x^*, Tx^*)] + \gamma[p(x_n, Tx^*) + p(x^*, Tx_n)], \end{aligned}$$

hence

$$\begin{aligned} p(x^*, Tx^*) \le p(x^*, x_{n+1}) + \alpha p(x_n, x^*) + \beta[p(x_n, x_{n+1}) + p(x^*, Tx^*)] \\ + \gamma[p(x_n, Tx^*) + p(x_{n+1}, x^*)] \end{aligned}$$

for all n. Taking the limit as $n \to \infty$, we get $p(x^*, Tx^*) \le (\beta + \gamma)\, p(x^*, Tx^*)$. Note that $\beta + \gamma < 1$, and so $p(x^*, Tx^*) = 0$. With $p(x^*, Tx^*) = 0 = p(x^*, x^*)$, implying, using (2.1) in [1], that $x^* \in \overline{Tx^*} = Tx^*$, since the mapping is into closed and bounded sets.

For the case $r = 0$: Note that the hypothesis gives $H_p(Tx, Ty) = 0$ for all $x, y \in X$. Hence, $Tx = Ty = A \neq \emptyset$, and any element a in A is a fixed point. □

The following corollaries then follow:

Corollary 3.3. *(Aydi [1]; Nadler [6]) Let (X, p) be a complete partial metric space and let $T : X \to CB^p(X)$ be a multi-valued mapping such that for all $x, y \in X, H_p(Tx, Ty) \le \alpha p(x, y)$ where $0 \le \alpha < 1$. Then T has a fixed point.*

Corollary 3.4. *Let (X, p) be a complete partial metric space and let $T : X \to CB^p(X)$ be a multi-valued mapping such that for all $x, y \in X$,*

$$H_p(Tx, Ty) \le \beta[p(x, Tx) + p(y, Ty)],$$

with $0 \le \beta < 1/2$. Then T has a fixed point.

Corollary 3.5. *Let (X, p) be a complete partial metric space and let $T : X \to CB^p(X)$ be a multi-valued mapping such that for all $x, y \in X$,*

$$H_p(Tx, Ty) \le \gamma[p(x, Ty) + p(y, Tx)]$$

with $0 \le \gamma < 1/2$. Then T has a fixed point.

Corollary 3.6. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a map from X into itself such that for all $x, y \in X$,*

$$p(Tx, Ty) \le \alpha p(x, y) + \beta[p(x, Tx) + p(y, Ty)] + \gamma[p(x, Ty) + p(y, Tx)],$$

$\alpha, \beta, \gamma \ge 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.

Further generalizations, using other conditions (parallel to the single-valued case) substituting for the contraction criterion, will be investigated, particularly for functions on partial metric spaces.

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