Int. Journal of Math. Analysis, Vol. 7, 2013, no. 8, 373 - 377

On the Multiple Sums of Bernoulli, Euler and Genocchi Polynomials

Burak Kurt

Akdeniz University, Faculty of Educations
Department of Mathematics, 07058-Antalya, Turkey
burakkurt@akdeniz.edu.tr

Abstract

We introduce and investigate the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials by means of a suitable theirs generating polynomials. We establish several interesting properties of these polynomials. Also, we gave some propositions two theorems and one corollary.

Keywords: Bernoulli polynomials, Euler polynomials, Genocchi polynomials, Apostol-Bernoulli polynomials, Apostol-Euler polynomials, Apostol-Genocchi polynomials, Generalized Hurwitz-Lerch zeta function

1 Introduction

It is well known that the Bernoulli polynomials $B_n(x)$, Euler polynomials $E_n(x)$, Genocchi polynomials $G_n(x)$ are of fundamental importance in several parts of analysis and in the calculus of finite differences and have applications in various other fields such as statistic, numerical analysis, combinatorics and so on.

Some interesting generalization of the classical Bernoulli polynomials and the classical Euler polynomials were investigated by Luo [5], Luo *et al.* [6], Srivastava *et al.* [2], [8], [10]. Also, Ozden *et al.* [8], Kurt [3], Kurt [4] were worked different means. Further, Brychkov [1] gave different recurrences relation.

In this work, we gave proposition two theorems on the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials.

The α-order Apostol-Bernoulli polynomials, Apostol-Euler polynomials, Apostol-Genocchi polynomials are defined by [2], [5], [6], [10] respectively

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{t}{\lambda e^t-1}\right)^{\alpha},\ |t+\ln\lambda|<2\pi, \tag{1}$$

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{2}{\lambda e^t+1}\right)^{\alpha},\ |t+\ln\lambda|<\pi \tag{2}$$

and

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t-1}\right)^{\alpha},\ |t+\ln\lambda|<\pi \tag{3}$$

where α reel or complex parameters; $\lambda\in\mathbb{C}$.

For $x=0$, we have Apostol-Bernoulli numbers $B_n^{(\alpha)}(\lambda)$, Apostol-Euler numbers $E_n^{(\alpha)}(\lambda)$, Apostol-Genocchi numbers $G_n^{(\alpha)}(\lambda)$ respectively.

Some of these Apostol-Bernoulli numbers, Apostol-Euler numbers and Apostol-Genocchi numbers are given respectively.

For $\alpha=1$,

$B_0(\lambda)=0$, $B_1(\lambda)=\frac{1}{\lambda-1}$, $B_2(\lambda)=\frac{-2\lambda}{(\lambda-1)^2}$, $B_3(\lambda)=\frac{3\lambda(\lambda-1)}{(\lambda-1)^3},\cdots,$

$E_0(\lambda)=\frac{2}{\lambda+1}$, $E_1(\lambda)=\frac{-2\lambda}{(\lambda+1)^2}$, $E_2(\lambda)=\frac{-2\lambda(\lambda-1)}{(\lambda+1)^3}$, $E_3(\lambda)=\frac{-2\lambda\left(\lambda^2-4\lambda+1\right)}{(\lambda+1)^4},\cdots$

and

$G_0(\lambda)=\frac{2}{\lambda+1}$, $G_1(\lambda)=\frac{2\lambda}{(\lambda+1)}$, $G_2(\lambda)=\frac{-4\lambda}{(\lambda+1)^2}$, $G_3(\lambda)=\frac{6\lambda(\lambda-1)}{(\lambda+1)^3},\cdots.$

We can obtain these numbers from 1, 2 and 3 easily.

Let us given power series

$$f_i(z)=\sum_{k=0}^{\infty} a_k^{(i)} z^k. \tag{4}$$

Then

$$\prod_{i=1}^{m} f_i(z)=\sum_{k=0}^{\infty} b_k z^k$$

where

$$b_k=\sum_{k_1+\cdots+k_m=k}\prod_{i=1}^{m}\frac{a_{k_i}^{(i)}}{k_i!}.$$

Note that, $B_k^{(1)}(z,\lambda)=B_k(z,\lambda)$, $E_k^{(1)}(z,\lambda)=E_k(z,\lambda)$ and $G_k^{(1)}(z,\lambda)=G_k(z,\lambda)$.

2 Main Theorems

Lemma 1 *The following relationships holds true;*

$$\sum_{k_1+\cdots+k_n=n}^{\infty}\prod_{i=1}^{m}\frac{1}{k_i!}B_{k_i}^{(\alpha)}(x_i,\lambda)=\frac{1}{n!}B_{n}^{(\alpha)}(x,\lambda), \tag{5}$$

$$\sum_{k_1+\cdots+k_n=n}^{\infty}\prod_{i=1}^{m}\frac{1}{k_i!}E_{k_i}^{(\alpha)}(x_i,\lambda)=\frac{1}{n!}E_{n}^{(\alpha)}(x,\lambda) \tag{6}$$

and

$$\sum_{k_1+\cdots+k_n=n}^{\infty}\prod_{i=1}^{m}\frac{1}{k_i!}G_{k_i}^{(\alpha)}(x_i,\lambda)=\frac{1}{n!}G_{n}^{(\alpha)}(x,\lambda) \tag{7}$$

where $\alpha=\alpha_1+\alpha_2+\cdots+\alpha_m$, $z=z_1+z_2+\cdots+z_m$.

Proof. From equation 1, we obtain

$$\begin{aligned}\prod_{i=1}^{m}\left[\sum_{k=0}^{\infty}B_{k}^{(\alpha)}(z_i,\lambda)\right] &= \prod_{i=1}^{m}\left(\frac{t}{\lambda e^t-1}\right)^{\alpha_i}e^{tz_i}\\ &= \frac{t^{\alpha}}{(\lambda e^t-1)^{\alpha}}e^{tz}\\ &= \sum_{k=0}^{\infty}B_{k}^{(\alpha)}(z,\lambda)\frac{t^k}{k!}.\end{aligned}$$

Comparing the coefficients of the first and last members, we get 5. ■

Lemma 2 *There are following equalities between Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials as*

1. $\displaystyle\sum_{k=0}^{n}\binom{n}{k}B_{k-n}^{(\alpha_1)}(x_1,\lambda)B_{k}^{(\alpha_2)}(x_2,\lambda)=B_{k}^{(\alpha_1+\alpha_2)}(x_1+x_2,\lambda)$

2. $\displaystyle\sum_{l=0}^{n}\binom{n}{l}B_{n}^{(\alpha)}(x,\lambda)E_{n-l}^{(\alpha)}(y,\lambda)=B_{n}^{(\alpha)}\left(x+y,\lambda^2\right)$

3. $\displaystyle\sum_{k=0}^{n}\binom{n}{k}B_{k}^{(\alpha)}(x,\lambda)G_{n-k}^{(\alpha)}(y,\lambda)=2^{n-\alpha}B_{n-\alpha}^{(\alpha)}\left(\frac{x+y}{2},\lambda^2\right)\binom{n}{n-\alpha}\alpha$

4. $$\sum_{k=0}^{n}\binom{n}{k}B_{k}^{(\alpha)}\left(x,\lambda\right)E_{n-k}^{(\alpha)}\left(y,\lambda\right)=2^{n}B_{n}^{(\alpha)}\left(\frac{x+y}{2},\lambda^{2}\right)$$

The generalized Hurwitz-Lerch zeta function is defined as

$$\phi_{m}^{*}\left(z,s,a\right)=\sum_{n=0}^{\infty}\frac{\left(\mu\right)_{n}}{n!}\frac{z^{n}}{\left(n+a\right)^{s}} \tag{8}$$

see [2], [4] and [10].

Theorem 3 *There is a relation between Apostol-Genocchi polynomials and generalized Hurwitz-Lerch zeta function as*

$$\begin{aligned} G_{n+l}^{(\alpha)}\left(x,\lambda\right) &= 2^{l}\binom{m+l}{l}l!\sum_{m=0}^{\infty}\binom{m+l-1}{m}\frac{\lambda^{m}}{\left(m+x\right)^{-n}} \\ &= 2^{l}\binom{m+l}{l}l!\phi_{m}^{*}\left(\lambda,-n,x\right). \end{aligned}$$

Theorem 4 *The Apostol-Bernoulli polynomials are satisfied the following relation*

$$B_{n}^{(\alpha)}\left(x,\lambda^{2}\right)=2^{-n}e^{-x\ln\lambda}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{(\alpha)}\left(x,\lambda\right)E_{k+p}^{(\alpha)}(x)\frac{\left(\ln\lambda\right)^{k}}{k!}.$$

Acknowledgement. This paper was supported by the Scientific Research Fund of Project Administration of Akdeniz University.

References

[1] Yu.A. Brychkov, On multiple sums of special functions, Integral Transforms and Special Functions Vol. 21(12) 2010, 877-884.

[2] M. Garg, K. Jain and H. M. Srivastava, Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch zeta functions, Integral Transforms Spec. Funct. 17 (2006), no. 11, 803-815.

[3] V. Kurt , A further symmetric relation on the analogue of the Apostol-Bernoulli and the analogue of the Apostol-Genocchi polynomials, Appl. Math. Sci. (Ruse) 3 (2009), 2757-2764.

[4] B. Kurt, The multiplication formulae for the Genocchi polynomials, Proc. Jangjeon Math. Soc. 13 (2010), 89–96.

[5] Q.-M. Luo, The multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials of higher order, Integral Transforms Spec. Funct. 20 (2009)(5-6), 377-391.

[6] Q.-M. Luo and H. M. Srivastava, Some generalization of the Apostol-Genocchi polynomials and Stirling numbers of the second kind, Appl. Math. and Comp. 217 (2011), no. 12, 5702–5728.

[7] Da-Qian Lu, Some properties of Bernoulli polynomials and their generalizations, Appl. Math. letter, 24(5), 2011, 746-751.

[8] H. Ozden, Y. Simsek and H. M. Srivastava, A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials, Comput. Math. Appl. 60 (2010), no. 10, 2779–2787.

[9] H. M. Srivastava, M. Garg and S. Choudhary, A new generalization of the Bernoulli and related polynomials, Russ. J. Math. Phys. (2010), (17)2, 251–261.

[10] H. M. Srivastava, M. Garg and S. Choudhary, Some New Families of Generalized Euler and Genocchi Polynomials, Tawiwanese J. Math. 15(2011) no.1, 283-305.

Received: September, 2012