Fixed Point Approach to the Estimation of Approximate General Quadratic Mappings

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Abstract

The purpose of this paper is to investigate the stability of a mixed functional equation with quadratic and Jensen additive mappings by using the fixed point method without dividing an approximate mapping by two approximate even and odd parts of the approximate mapping.

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1 Introduction

In 1940, S. M. Ulam [27] raised a question concerning the stability of homomorphisms: Given a group $G_1$, a metric group $G_2$ with the metric $d(\cdot, \cdot)$, and a positive number $\varepsilon$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \rightarrow G_2$
satisfies the inequality \( d(f(xy), f(x)f(y)) < \delta \) for all \( x, y \in G_1 \), then there exists a homomorphism \( h : G_1 \to G_2 \) with \( d(f(x), h(x)) < \varepsilon \) for all \( x \in G_1 \)? When this problem has a solution, we say that the homomorphisms from \( G_1 \) to \( G_2 \) are stable. Thus, for very general functional equations, the stability question of functional equations is that a functional equation \( \mathcal{F}_1(f) = \mathcal{F}_2(f) \) is stable if any function \( f \) satisfying approximately the equation \( \mathcal{F}_1(f) = \mathcal{F}_2(f) \) is near to an exact solution \( h \) of the equation \( \mathcal{F}_1(f) = \mathcal{F}_2(f) \). In 1941, D. H. Hyers [11] considered the case of approximately additive mappings between Banach spaces and proved the following result. Suppose that \( E_1 \) and \( E_2 \) are Banach spaces and \( f : E_1 \to E_2 \) satisfies the following condition: there is a constant \( \epsilon \geq 0 \) such that
\[
\| f(x + y) - f(x) - f(y) \| \leq \epsilon
\]
for all \( x, y \in E_1 \). Then the limit \( h(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \) exists for all \( x \in E_1 \) and it is a unique additive mapping \( h : E_1 \to E_2 \) such that \( \| f(x) - h(x) \| \leq \epsilon \).

The method which was provided by Hyers, and which produces the additive mapping \( h \), was called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations. Hyers' Theorem was generalized by T. Aoki [1] and D.G. Bourgin [3] for additive mappings by considering an unbounded Cauchy difference. In 1978, Th.M. Rassias [22] also provided a generalization of Hyers' Theorem for linear mappings which allows the Cauchy difference to be unbounded like this \( \| x \|^p + \| y \|^p \). Let \( E_1 \) and \( E_2 \) be two Banach spaces and \( f : E_1 \to E_2 \) be a mapping such that \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \). Assume that there exists \( \epsilon > 0 \) and \( 0 \leq p < 1 \) such that
\[
\| f(x + y) - f(x) - f(y) \| \leq \epsilon(\| x \|^p + \| y \|^p), \quad \forall x, y \in E_1.
\]
Then there exists a unique \( \mathbb{R} \)-linear mapping \( T : E_1 \to E_2 \), defined by \( T(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \) such that
\[
\| f(x) - T(x) \| \leq \frac{2\epsilon}{2 - 2^p} \| x \|^p
\]
for all \( x \in E_1 \). A generalized result of Th.M. Rassias' theorem was obtained by P. Gavruta in [9] and S. Jung in [14]. In 1990, Th.M. Rassias [23] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for \( p \geq 1 \). In 1991, Z. Gajda [8] following the same approach as in [22], gave an affirmative solution to this question for \( p > 1 \). It was shown by Z. Gajda [8], as well as by Th.M. Rassias and P. Šemrl [24], that one cannot prove a Th.M. Rassias' type theorem when \( p = 1 \). The counterexamples of Z. Gajda [8], as well as of Th.M. Rassias and P. Šemrl [24], have stimulated several mathematicians to invent new approximately additive or approximately linear mappings. In particular, J.M. Rassias
[20, 21] proved a similar stability theorem in which he replaced the unbounded Cauchy difference by this factor $\|x\|^p \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

In 2003, L. Cădariu and V. Radu [4] observed that the existence of the solution $F$ for a Cauchy functional equation and the estimation of the mapping $F$ with the approximate mapping $f$ of the equation can be obtained from the alternative fixed point theorem. This method is called a fixed point method. In addition, they applied this method to prove the stability theorems of the Jensen’s functional equation:

$$2f \left( \frac{x+y}{2} \right) - f(x) - f(y) = 0 \Leftrightarrow 2f(x) - f(x+y) - f(x-y) = 0. \quad (1)$$

Furthermore, they [4] obtained the stability of the quadratic functional equation:

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0 \quad (2)$$

by using the fixed point method. There are many interesting stability results concerning the stability problems obtained by using the fixed point method. A large list of references can be found in the papers [5, 6, 7, 10, 12, 13, 16, 17, 19] and references therein.

If we consider the functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = ax + b$ and $f_2(x) = cx^2$, respectively, where $a, b$ and $c$ are real constants, then we notice that $f_1$ satisfies the equation (1) and $f_2$ is a solution of the equation (2), respectively. In 2008, A. Najati and M. Moghimi [18] obtained stability results of the following functional equation

$$2f(x+y) + f(x-2y) + 3f(y) - 3f(x) - f(-y) - 2f(2y) = 0, \quad (3)$$

of which the general solution function $f$ between linear spaces is of the form $f(x) = Q(x) + A(x) + f(0)$, where $Q(x)$ is a quadratic mapping satisfying the equation (2) and $A(x) + f(0)$ is a Jensen mapping satisfying the equation (1). Thus, we say that the equation (3) a general quadratic functional equation and a solution of (3) is called a general quadratic mapping. In their processing [18], by splitting to handle the odd part and the even part of the given mapping $f$, respectively, they could have an additive mapping $A$ which is close to the odd part $\frac{f(x) - f(-x)}{2}$ of $f$ and a quadratic mapping $Q$ which is approximate to the even part $\frac{f(x) + f(-x)}{2} - f(0)$ of it, and then they could prove the existence of a general quadratic mapping $F$ which is close to the given mapping $f$ by combining $A$ and $Q + f(0)$.

Now, without splitting the given approximate mapping $f : X \to Y$ of the equation (3) into two approximate even and odd parts, we are going to derive the desired solution $F$ near the approximate mapping $f$ in this paper. Precisely, we introduce a strictly contractive mapping with Liptshitz constant.
0 < L < 1. And then, using the fixed point method in the sense of L. Cădariu and V. Radu together with suitable conditions, we can show that the contractive mapping has the fixed point $F$ in a generalized metric function space. Actually, the fixed point $F$ yields the precise solution of the functional equation (3) near $f$. In Section 2, we prove several stability results of the functional equation (3) using the fixed point method under suitable conditions. In Section 3, we use the results in Section 2 to get stability result of the Jensen’s functional equation (1) and to get that of the quadratic functional equation (2), respectively.

2 generalized Hyers-Ulam stability of (3)

In this section, we prove the generalized Hyers–Ulam stability of the general quadratic functional equation (3). We recall the following fundamental result of the fixed point theory [15, 26].

**Theorem 2.1** Suppose that a complete generalized metric space $(X, d)$, which means that the metric $d$ may assume infinite values, and a strictly contractive mapping $\Lambda : X \rightarrow X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either

$$d(\Lambda^n x, \Lambda^{n+1} x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\},$$

or there exists a nonnegative integer $k$ such that:

- $d(\Lambda^n x, \Lambda^{n+1} x) < +\infty$ for all $n \geq k$;
- the sequence $\{\Lambda^n x\}$ is convergent to a fixed point $y^*$ of $\Lambda$;
- $y^*$ is the unique fixed point of $\Lambda$ in $X_1 := \{y \in X, d(\Lambda^k x, y) < +\infty\}$;
- $d(y, y^*) \leq \frac{1}{1-L} d(y, \Lambda y)$ for all $y \in X_1$.

Throughout this paper, let $V$ be a linear space and $Y$ a Banach space. For a given mapping $f : V \rightarrow Y$, we use the following abbreviation

$$Df(x, y) := 2f(x + y) + f(x - 2y) + 3f(y) - 3f(x) - f(-y) - 2f(2y)$$

for all $x, y \in V$.

In the following theorem, we can prove the stability of the general quadratic functional equation (3) using the fixed point method.
Theorem 2.2 Let $f : V \rightarrow Y$ be a mapping for which there exists a mapping $\varphi : V^2 \rightarrow [0, \infty)$ such that
\[ \|Df(x, y)\| \leq \varphi(x, y) \] (4)
for all $x, y \in V$. If $\varphi(x, y) = \varphi(-x, -y)$ for all $x, y \in V$ and there exists a constant $0 < L < 1$ such that
\[ \varphi(2x, 2y) \leq 2L \varphi(x, y), \] (5)
for all $x, y \in V$, then there exists a unique mapping $F : V \rightarrow Y$ such that $DF(x, y) = 0$ for all $x, y \in V$ and
\[ \|f(x) - F(x)\| \leq \frac{\varphi(0, x)}{4(1 - L)} \] (6)
for all $x \in V$. In particular, $F$ is represented by
\[ F(x) = f(0) + \lim_{n \rightarrow \infty} \left[ \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \right] \] (7)
for all $x \in V$.

Proof. It follows from (5) that
\[ \lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y)}{2^n} \leq \lim_{n \rightarrow \infty} L^n \varphi(x, y) = 0 \]
for all $x, y \in V$. If we consider the mapping $\tilde{f} := f - f(0)$, then $\tilde{f} : V \rightarrow Y$ satisfies $\tilde{f}(0) = 0$ and
\[ \|D\tilde{f}(x, y)\| = \|Df(x, y)\| \leq \varphi(x, y) \] (8)
for all $x, y \in V$. Let $S$ be the set of all mappings $g : V \rightarrow Y$ with $g(0) = 0$, and then we introduce a generalized metric $d$ on $S$ by
\[ d(g, h) := \inf \{ K \in \mathbb{R}^+ \| g(x) - h(x) \| \leq K \varphi(0, x) \ \forall x \in V \}. \] (9)
It is easy to show that $(S, d)$ is a generalized complete metric space. Now, we consider the mapping $\Lambda : S \rightarrow S$ defined by
\[ \Lambda g(x) := \frac{g(2x) - g(-2x)}{4} + \frac{g(2x) + g(-2x)}{8} \] (10)
for all $g \in S$ and all $x \in V$. Then we notice that
\[ \Lambda^n g(x) = \frac{g(2^n x) - g(-2^n x)}{2^{n+1}} + \frac{g(2^n x) + g(-2^n x)}{2 \cdot 4^n} \] (11)
for all \( n \in \mathbb{N} \) and \( x \in V \).

First, we show that \( \Lambda \) is a strictly contractive self-mapping of \( S \) with the Lipschitz constant \( L \). Let \( g, h \in S \) and let \( K \in [0, \infty) \) be an arbitrary constant with \( d(g, h) \leq K \). From the definition of \( d \), we have

\[
\|\Lambda g(x) - \Lambda h(x)\| = \frac{3}{8}\|g(2x) - h(2x)\| + \frac{1}{8}\|g(-2x) - h(-2x)\| \leq \frac{1}{2}K\varphi(0, 2x) \leq LK\varphi(0, x)
\]

for all \( x \in V \), which implies that

\[
d(\Lambda g, \Lambda h) \leq Ld(g, h)
\]

for any \( g, h \in S \). That is, \( \Lambda \) is a strictly contractive self-mapping of \( S \) with the Lipschitz constant \( L \). Moreover, by (8) we see that

\[
\|\tilde{f}(x) - \Lambda \tilde{f}(x)\| = \frac{1}{24}\|5D\tilde{f}(0, x) + D\tilde{f}(0, -x)\| \leq \frac{1}{4}\varphi(0, x)
\]

for all \( x \in V \). It means that \( d(\tilde{f}, \Lambda \tilde{f}) \leq \frac{1}{4} < \infty \) by the definition of \( d \).

Therefore, according to Theorem 2.1, the sequence \( \{\Lambda^n \tilde{f}\} \) converges to the fixed point \( \tilde{F}: V \to Y \) of \( \Lambda \) satisfying

\[
\tilde{F}(x) = \frac{\tilde{F}(2x) - \tilde{F}(-2x)}{4} + \frac{\tilde{F}(2x) + \tilde{F}(-2x)}{8},
\]

which is represented by

\[
\tilde{F}(x) = \lim_{n \to \infty} \Lambda^n \tilde{f}(x) = \lim_{n \to \infty} \left[ \frac{\tilde{f}(2^n x) - \tilde{f}(-2^n x)}{2^{n+1}} + \frac{\tilde{f}(2^n x) + \tilde{f}(-2^n x)}{2 \cdot 4^n} \right]
\]

for all \( x \in V \). In addition, it is unique in the set \( S_1 = \{g \in S | d(\tilde{f}, g) < \infty \} \).

Putting \( F(x) := f(0) + \tilde{F}(x) \), then one notes that the equality \( \|f(x) - F(x)\| = \|\tilde{f}(x) - \tilde{F}(x)\| \) for all \( x \in V \) and

\[
d(\tilde{f}, F) \leq \frac{1}{1 - L}d(\tilde{f}, \Lambda \tilde{f}) \leq \frac{1}{4(1 - L)},
\]

which implies (6) and (7).

By the definitions of \( F \) and \( \tilde{F} \), together with (8) and (5), we have that

\[
\|DF(x, y)\| = \|D\tilde{F}(x, y)\| = \lim_{n \to \infty} \left\| Df(2^n x, 2^n y) - f(-2^n x, -2^n y) \right\|
\]

\[
= \lim_{n \to \infty} \left\| Df\left(\frac{2^{n+1}}{2} \cdot 4^n \right) \frac{Df(2^n x, 2^n y) + Df(-2^n x, -2^n y)}{2 \cdot 4^n} \right\|
\]

\[
\leq \lim_{n \to \infty} \frac{2^n + 1}{2 \cdot 4^n} \left( \varphi(2^n x, 2^n y) + \varphi(-2^n x, -2^n y) \right)
\]

\[
= 0
\]
for all \( x, y \in V \). Thus, the mapping \( F \) satisfies the general quadratic functional equation (6). This completes the proof of this theorem.

**Theorem 2.3** Let \( f : V \rightarrow Y \) be a mapping for which there exists a mapping \( \varphi : V^2 \rightarrow [0, \infty) \) such that

\[
\| Df(x, y) \| \leq \varphi(x, y)
\]

for all \( x, y \in V \). If \( \varphi(x, y) = \varphi(-x, -y) \) for all \( x, y \in V \) and there exists a constant \( 0 < L < 1 \) such that the mapping \( \varphi \) has the property

\[
L \varphi(2x, 2y) \geq 4 \varphi(x, y)
\]

for all \( x, y \in V \), then there exists a unique mapping \( F : V \rightarrow Y \) such that

\[
DF(x, y) = 0
\]

for all \( x, y \in V \) and

\[
\| f(x) - F(x) \| \leq \frac{L \varphi(0, x)}{4(1 - L)}
\]

for all \( x \in V \). In particular, \( F \) is represented by

\[
F(x) := f(0) + \lim_{n \to \infty} \frac{4^n}{2} \left( f\left(\frac{x}{2^n}\right) + f\left(\frac{-x}{2^n}\right) - 2f(0)\right) + 2^{n-1} \left( f\left(\frac{x}{2^n}\right) - f\left(\frac{-x}{2^n}\right)\right)
\]

for all \( x \in V \).

**Proof.** If we consider the mapping \( \tilde{f} = f - f(0) \), then \( \tilde{f} : V \rightarrow Y \) satisfies \( \tilde{f}(0) = 0 \) and

\[
\| D\tilde{f}(x, y) \| = \| Df(x, y) \| \leq \varphi(x, y)
\]

for all \( x, y \in V \). Let the space \((S, d)\) be as in the proof of Theorem 2.2. Now, we consider a mapping \( \Lambda : S \rightarrow S \) defined by

\[
\Lambda g(x) := g\left(\frac{x}{2}\right) - g\left(-\frac{x}{2}\right) + 2\left(g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right)\right)
\]

for all \( g \in S \) and \( x \in V \). Then we observe that

\[
\Lambda^n g(x) = 2^{n-1} \left( g\left(\frac{x}{2^n}\right) - g\left(-\frac{x}{2^n}\right)\right) + \frac{4^n}{2} \left( g\left(\frac{x}{2^n}\right) + g\left(-\frac{x}{2^n}\right)\right)
\]

and \( \Lambda^0 g(x) = g(x) \) for all \( x \in V \). Let \( g, h \in S \) and let \( K \in [0, \infty] \) be an arbitrary constant with \( d(g, h) \leq K \). From the definition of \( d \), we have

\[
\| \Lambda g(x) - \Lambda h(x) \| \leq 3\| g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right) \| + \| g\left(-\frac{x}{2}\right) - h\left(-\frac{x}{2}\right) \|
\]

\[
\leq 4K \varphi\left(\frac{x}{2}\right) \leq LK \varphi(0, x)
\]
for all \( x \in V \). So
\[
d(\Lambda g, \Lambda h) \leq Ld(g, h)
\]
for any \( g, h \in S \). That is, \( \Lambda \) is a strictly contractive self-mapping of \( S \) with the Lipschitz constant \( L \). Also we see that
\[
\| \tilde{f}(x) - \Lambda \tilde{f}(x) \| = \frac{1}{3} \| -2D\tilde{f}(0, \frac{x}{2}) - D\tilde{f}(0, -\frac{x}{2}) \| \\
\leq \varphi(0, \frac{x}{2}) \leq \frac{L}{4} \varphi(0, x)
\]
for all \( x \in V \), which implies that \( d(\tilde{f}, \Lambda \tilde{f}) \leq \frac{L}{4} < \infty \). Therefore according to Theorem 2.1, the sequence \( \{\Lambda^n \tilde{f}\} \) converges to a fixed point \( \tilde{F} \) of \( \Lambda \) in the set \( S_1 := \{g \in S|d(f, g) < \infty\} \), that is,
\[
\tilde{F}(x) = \tilde{F}(\frac{x}{2}) - \tilde{F}(-\frac{x}{2}) + 2(\tilde{F}(\frac{x}{2}) + \tilde{F}(-\frac{x}{2}))
\]
which is represented by
\[
\tilde{F}(x) = \lim_{n \to \infty} \left[2^{n-1}(\tilde{f}(\frac{x}{2^n}) - \tilde{f}(-\frac{x}{2^n})) + \frac{4^n}{2}(\tilde{f}(\frac{x}{2^n}) + \tilde{f}(-\frac{x}{2^n}))\right]
\]
for all \( x \in V \). Putting \( F(x) := \tilde{F}(x) + f(0), x \in V \), we then know that \( \|f(x) - F(x)\| = \|\tilde{f}(x) - \tilde{F}(x)\| \) for all \( x \in V \) and
\[
d(\tilde{f}, F) \leq \frac{1}{1-L}d(\tilde{f}, \Lambda \tilde{f}) \leq \frac{L}{4(1-L)},
\]
which implies (19). From the definition of \( F \) and \( \tilde{F} \), together with (18), and (21), we have
\[
\|DF(x, y)\| = \|D\tilde{F}(x, y)\|
\]
\[
\leq \lim_{n \to \infty} \left[2^{n-1}\left(Df(\frac{x}{2^n}, \frac{y}{2^n}) - Df(-\frac{x}{2^n}, -\frac{y}{2^n})\right) \right.
\left.+ \frac{4^n}{2}\left(Df(\frac{x}{2^n}, \frac{y}{2^n}) + Df(-\frac{x}{2^n}, -\frac{y}{2^n})\right)\right]
\]
\[
\leq \lim_{n \to \infty} \left(2^{n-1} + \frac{4^n}{2}\right)\left[\varphi(\frac{x}{2^n}, \frac{y}{2^n}) + \varphi(-\frac{x}{2^n}, -\frac{y}{2^n})\right]
\]
\[
= 0
\]
for all \( x, y \in V \) and so the mapping \( F \) satisfies the general quadratic functional equation (6). This completes the proof of this theorem.
3 Applications

For a given mapping \( f : V \to Y \), we use the following abbreviations
\[
Jf(x, y) := 2f\left(\frac{x + y}{2}\right) - f(x) - f(y),
\]
\[
Qf(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y)
\]
for all \( x, y \in V \). Using Theorem 2.2 and Theorem 2.3, we will obtain the stability results of the Jensen’s functional equation \( Jf \equiv 0 \) and the quadratic functional equation \( Qf \equiv 0 \) in the following corollaries.

**Corollary 3.1** Let \( f_i : V \to Y, \ i = 1, 2 \), be mappings for which there exist functions \( \phi_i : V^2 \to [0, \infty), \ i = 1, 2 \), such that
\[
\|Jf_i(x, y)\| \leq \phi_i(x, y)
\]
and \( \phi_i(x, y) = \phi_i(-x, -y) \) for all \( x, y \in V \), respectively. If there exists \( 0 < L < 1 \) such that
\[
\phi_1(2x, 2y) \leq 2L\phi_1(x, y), \quad (23)
\]
\[
L\phi_2(2x, 2y) \geq 4\phi_2(x, y) \quad (24)
\]
for all \( x, y \in V \), then there exist unique Jensen mappings \( F_i : V \to Y, \ i = 1, 2 \), such that
\[
\|f_1(x) - F_1(x)\| \leq \frac{\varphi_1(0, x)}{4(1 - L)}, \quad (25)
\]
\[
\|f_2(x) - F_2(x)\| \leq \frac{L\varphi_2(0, x)}{4(1 - L)} \quad (26)
\]
for all \( x \in V \), where \( \varphi_i, i = 1, 2 \), are defined by
\[
\varphi_i(x, y) := \phi_i(-y, y) + 2\phi_i(2y, 0) + \phi_i(2x, 2y) + \frac{1}{2}\phi_i(2x, -4y)
\]
\[
+ \frac{1}{2}\phi_i(0, -4y) + \phi_i(2y, -2y) + \frac{3}{2}\phi_i(2x, 0).
\]

In particular, the mappings \( F_1, F_2 \) are represented by
\[
F_1(x) = \lim_{n \to \infty} \frac{f_1(2^n x)}{2^n} + f_1(0), \quad (27)
\]
\[
F_2(x) = \lim_{n \to \infty} 2^n \left( f_2\left(\frac{x}{2^n}\right) - f_2(0)\right) + f_2(0) \quad (28)
\]
for all \( x \in V \).
Proof. First, we observe from (22) that

\[ \|Df_i(x, y)\| = \|Jf_i(-y, y) + 2Jf_i(2y, 0) + Jf_i(2x, 2y) + \frac{1}{2}Jf_i(2x, -4y) - \frac{1}{2}Jf_i(0, -4y) - Jf_i(2y, -2y) - \frac{3}{2}Jf_i(2x, 0)\| \leq \varphi_i(x, y), \]

where we recall the following notation

\[ Df_i(x, y) := 2f_i(x + y) + f_i(x - 2y) + 3f_i(y) - 3f_i(x) - f_i(-y) - 2f_i(2y) \]

for all \( x, y \in V \) and \( i = 1, 2 \). Since \( \varphi_1 \) satisfies (5), according to Theorem 2.2, there exists a unique mapping \( F_1 : V \to Y \) satisfying the desired approximation (25), which is represented by (7).

We observe from (22) and (23) that

\[
\lim_{n \to \infty} \left\| \frac{f_i(2^n x) + f_i(-2^n x)}{2^{n+1}} \right\| = \lim_{n \to \infty} \left\| \frac{f_i(2^n x) + f_i(-2^n x) - 2f_i(0)}{2^{n+1}} \right\| = \lim_{n \to \infty} \frac{1}{2^{n+1}} \|Jf_i(2^n x, -2^n x)\| \\
\leq \lim_{n \to \infty} \frac{1}{2^{n+1}} \phi_1(2^n x, -2^n x) \leq \lim_{n \to \infty} \frac{L^n}{2} \phi_1(x, -x) = 0
\]

as well as

\[
\lim_{n \to \infty} \left\| \frac{f_1(2^n x) + f_1(-2^n x)}{2 \cdot 4^n} \right\| \leq \lim_{n \to \infty} \frac{2^n L^n}{2 \cdot 4^n} \phi_1(x, -x) = 0
\]

for all \( x \in V \). From these properties and the representation (7), we get the mapping \( F_1 : V \to Y \) defined as (27). Moreover, we have

\[
\left\| \frac{Jf_1(2^n x, 2^n y)}{2^n} \right\| \leq \frac{\phi_1(2^n x, 2^n y)}{2^n} \leq L^n \phi_1(x, y)
\]

for all \( x, y \in V \). Thus, taking the limit as \( n \to \infty \) in the above inequality, we get \( JF_1(x, y) = 0 \) for all \( x, y \in V \) and so \( F_1 : V \to Y \) is a Jensen mapping.

On the other hand, we know that \( \varphi_2 \) satisfies (18). Therefore according to Theorem 2.3, there exists a unique mapping \( F_2 : V \to Y \) satisfying (26) which is represented by (20). Observe that

\[
\lim_{n \to \infty} 2^{2n-1} \left\| f_2\left(\frac{x}{2^n}\right) + f_2\left(\frac{-x}{2^n}\right) - 2f_2(0) \right\| = \lim_{n \to \infty} 2^{2n-1} \left\| Jf_2\left(\frac{x}{2^n}, -\frac{x}{2^n}\right) \right\| \\
\leq \lim_{n \to \infty} 2^{2n-1} \phi_2\left(\frac{x}{2^n}, -\frac{x}{2^n}\right) \\
\leq \lim_{n \to \infty} \frac{L^n}{2} \phi_2(x, -x) = 0
\]
as well as

\[
\lim_{n \to \infty} \frac{2^n}{2} \| f_2 \left( \frac{x}{2^n} \right) + f_2 \left( \frac{-x}{2^n} \right) - 2 f_2(0) \| = 0
\]

for all \( x \in V \). From these and (20), we get (28). Moreover, we have

\[
\| 2^n J f_2 \left( \frac{x}{2^n}, \frac{y}{2^n} \right) \| \leq 2^n \phi_2 \left( \frac{x}{2^n}, \frac{y}{2^n} \right) \leq \frac{L^n}{2^n} \phi_2(x, y)
\]

for all \( x, y \in V \). Taking the limit as \( n \to \infty \) in the above inequality, we get \( J F_2(x, y) = 0 \) for all \( x, y \in V \) and so \( F_2 : V \to Y \) is a Jensen mapping. This completes the corollary.

Now we obtain a stability result of quadratic functional equations.

**Corollary 3.2** Let \( \phi_i : V^2 \to [0, \infty), (i = 1, 2) \) be given function. Suppose that each \( f_i : V \to Y, (i = 1, 2) \) satisfies

\[
\| Q f_i(x, y) \| \leq \phi_i(x, y)
\]

for all \( x, y \in V \). If \( \phi_i, (i = 1, 2) \) satisfies the symmetrical property \( \phi_i(x, y) = \phi_i(-x, -y) \) and there exists a constant \( 0 < L < 1 \) such that the mapping \( \phi_1 \) satisfies the property (23) and \( \phi_2 \) satisfies (24) for all \( x, y \in V \), then we have unique quadratic mappings \( F_1, F_2 : V \to Y \) such that

\[
\| f_1(x) - f_1(0) - F_1(x) \| \leq \frac{\phi_1(0, 2x) + \phi_1(x, x) + \phi_1(0, x) + \phi_1(0, 0)}{4(1 - L)}, \quad (29)
\]

\[
\| f_2(x) - F_2(x) \| \leq \frac{L[\phi_2(0, 2x) + \phi_2(x, x) + \phi_2(0, x) + \phi_2(0, 0)]}{4(1 - L)} \quad (30)
\]

for all \( x \in V \), where \( F_1 \) and \( F_2 \) are represented by

\[
F_1(x) = \lim_{n \to \infty} \frac{f_1(2^n x)}{4^n}, \quad (31)
\]

\[
F_2(x) = \lim_{n \to \infty} 4^n f_2 \left( \frac{x}{2^n} \right) \quad (32)
\]

for all \( x \in V \). Moreover, if \( 0 < L < \frac{1}{2} \) and \( \phi_1 \) is continuous, then \( f_1 - f_1(0) \) is itself a quadratic mapping.

**Proof.** We note that

\[
\| D f_i(x, y) \| = \| Q f_i(x, 2y) - Q f_i(x + y, y) - Q f_i(0, y) + Q f_i(0, 0) \|
\]

\[
\leq \phi_i(x, 2y) + \phi_i(x + y, y) + \phi_i(0, y) + \phi_i(0, 0) =: \varphi_i(x, y)
\]

for all \( x, y \in V \) and \( i = 1, 2 \). Then \( \varphi_1 \) satisfies (5) and \( \varphi_2 \) satisfies (18). According to Theorem 2.2, there exists a unique mapping \( F_1 : V \to Y \) satisfying (29) which is represented by

\[
F_1(x) = \lim_{n \to \infty} \left( \frac{f_1(2^n x) + f_1(-2^n x)}{2 \cdot 4^n} + \frac{f_1(2^n x) - f_1(-2^n x)}{2^{n+1}} \right).
\]
Observe that
\[
\lim_{n \to \infty} \left\| \frac{f_1(2^n x) - f_1(-2^n x)}{2^{n+1}} \right\| = \lim_{n \to \infty} \frac{1}{2^{n+1}} \left\| Qf_1(0, 2^n x) \right\| \\
\leq \lim_{n \to \infty} \frac{1}{2^{n+1}} \phi_1(0, 2^n x) \\
\leq \lim_{n \to \infty} \frac{L^n}{2^n} \phi_1(0, x) = 0
\]
as well as
\[
\lim_{n \to \infty} \left\| \frac{f_1(2^n x) - f_1(-2^n x)}{2 \cdot 4^n} \right\| \leq \lim_{n \to \infty} \frac{L^n}{2^{n+1}} \phi_1(0, x) = 0
\]
for all \( x \in V \). From these, we lead to (31) for all \( x \in V \). Moreover, we have
\[
\left\| \frac{Qf_1(2^n x, 2^n y)}{4^n} \right\| \leq \frac{\phi_1(2^n x, 2^n y)}{4^n} \leq \frac{L^n}{2^n} \phi_1(x, y)
\]
for all \( x, y \in V \). Taking the limit as \( n \to \infty \) in the above inequality, we get
\[
QF_1(x, y) = 0
\]
for all \( x, y \in V \) and so \( F_1 : V \to Y \) is a quadratic mapping.

On the other hand, since \( L \phi_2(0, 0) \geq 4 \phi_2(0, 0) \)
\[
\|2f_2(0)\| = \|Qf_2(0, 0)\| \leq \phi_2(0, 0)
\]
we can show that \( \phi_2(0, 0) = 0 \) and \( f_2(0) = 0 \). According to Theorem 2.3, there exists a unique mapping \( F_2 : V \to Y \) satisfying (30), which is represented by (20). We have
\[
\lim_{n \to \infty} \frac{4^n}{2} \left\| f_2 \left( \frac{x}{2^n} \right) + f_2 \left( -\frac{x}{2^n} \right) \right\| = \lim_{n \to \infty} \frac{4^n}{2} \left\| Qf_2 \left( 0, \frac{x}{2^n} \right) \right\| \\
\leq \lim_{n \to \infty} \frac{4^n}{2} \phi_2 \left( 0, \frac{x}{2^n} \right) \\
\leq \lim_{n \to \infty} \frac{L^n}{2} \phi_2(0, x) = 0
\]
as well as
\[
\lim_{n \to \infty} 2^{n-1} \left\| f_2 \left( \frac{x}{2^n} \right) - f_2 \left( -\frac{x}{2^n} \right) \right\| = 0
\]
for all \( x \in V \). From this and (11), we get (32). Notice that
\[
\left\| 4^n Qf_2 \left( \frac{x}{2^n}, \frac{y}{2^n} \right) \right\| \leq 4^n \phi_2 \left( \frac{x}{2^n}, \frac{y}{2^n} \right) \leq L^n \phi_2(x, y)
\]
for all \( x, y \in V \). Taking the limit as \( n \to \infty \), then we have shown that

\[
QF_2(x, y) = 0
\]

for all \( x, y \in V \) and so \( F_2 : V \to Y \) is a quadratic mapping. This completes the corollary.

Now, we obtain generalized Hyers-Ulam stability results in the framework of normed spaces using Theorem 2.2 and Theorem 2.3.

**Corollary 3.3** Let \( X \) be a normed space, \( \theta \geq 0 \), and \( p \in (0,1) \cup (2,\infty) \). Suppose that a mapping \( f : X \to Y \) satisfies the inequality

\[
\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p)
\]

for all \( x, y \in X \). Then there exists a unique general quadratic mapping \( F : X \to Y \) satisfying \( DF(x, y) = 0 \) such that

\[
\|f(x) - F(x)\| \leq \begin{cases} 
\frac{\theta}{2(2-p^2)}\|x\|^p, & \text{if } 0 < p < 1; \\
\frac{\theta}{2p-1}\|x\|^p, & \text{if } p > 2,
\end{cases}
\]

for all \( x \in X \).

**Proof.** By putting \( \varphi(x,y) := \theta(\|x\|^p + \|y\|^p) \) together with \( L := 2^{p-1} < 1 \) if \( 0 < p < 1 \) and \( L := 2^{2-p} < 1 \) if \( p > 2 \), then we get the desired results by applying Theorem 2.2 and Theorem 2.3.

**Corollary 3.4** Let \( X \) be a normed space and \( p, q > 0 \) with \( p + q \in (0,1) \cup (2,\infty) \). If a mapping \( f : X \to Y \) satisfies

\[
\|Df(x, y)\| \leq \theta\|x\|^p\|y\|^q
\]

for all \( x, y \in X \), then \( f \) is itself a general quadratic mapping.

**Proof.** It follows from Theorem 2.2 and Theorem 2.3, by putting \( L := 2^{p+q-1} < 1 \) if \( p + q < 1 \) and \( L := 2^{2-p-q} < 1 \) if \( p + q > 2 \).

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Approximate general quadratic mappings


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