Recursive Summation of the $n$th Powers of Consecutive Congruent Numbers

P. Juntharee* and P. Prommi

Department of Mathematics
Faculty of Applied Science
King Mongkut’s University of Technology North Bangkok
1518 Phacharat 1 Road, Bangsue, Bangkok 10800, Thailand
pjt@kmutnb.ac.th, ptpm@kmutnb.ac.th

Abstract

The main purpose of this paper is to find a recursive form for the summation of $n$th powers of consecutive congruent numbers. Explicit summation formulas for the first, second and third powers of consecutive congruent numbers are derived. These formulas are applied to some problems involving the $n$th powers of consecutive congruent numbers.

Keywords: Recursive Summation, Consecutive Numbers, Congruent Numbers, Modulo

1 Introduction

Calculating sums of $n$th powers of consecutive congruent numbers can take considerable time. However, the sums can be easily obtained if a recursive form is available for the summation of $n$th powers of consecutive congruent numbers. With that motivation, we first revisit previous studies by Shiflett and Shultz [7] and Juntharee and Prommi [6]. Shiflett and Shultz showed that a positive number $n$ can be expressed as a sum of consecutive congruent numbers $2j + 1, 2j + 3, ..., 2s - 1$ for some integers $j$ and $s$ with $j \geq 0$ and $s > j + 1$ if and only if $n = (s - j)(s + j)$. On the other hand, Juntharee and Prommi [6] described a theorem giving conditions under which a positive number $n$ can be expressed as a sum of consecutive positive congruent numbers. It was shown that the required conditions are as follows:

For a given positive number $m$ and for $r = 0, 1, 2, 3, ..., m - 1$, a positive number $n$ can be expressed as a sum of consecutive positive congruent numbers which are congruent to $r$ modulo $m$ if and only if
case 1) if $m$ is an even positive number and $m = 2m_0$ for some positive number $m_0$ then $n$ can be factorized as $n = ab$ for some positive numbers $a, b$, where $a \leq b$ and $b = m_0(2i + a - 1) + r$ for some $i = 0, 1, 2, 3, ...$.

case 2) if $m$ is an odd positive number then $2n$ can be factorized as $2n = ab$ for some positive numbers $a, b$, where $a \leq b$ and $b = m(2i + a - 1) + 2r$ for some $i = 0, 1, 2, 3, ...$.

These results stimulated us to undertake this study. The outline of the present paper is as follows. In section 2, basic definitions and results required in this research are summarized. These include definitions of consecutive congruent numbers, and the binomial theorem, Pascal’s triangle and sums of a finite series. In section 3, a recursive form is derived for calculating the sums of $n$th powers of consecutive congruent numbers and explicit summation formulas are obtained for sums of first, second and third powers of consecutive congruent numbers. In section 4, the applications of the results to two practical problems are described and conclusions are made.

### 2 Preliminaries and Notation

In this section, we summarize some general mathematical background required in this work. For definition 2.1 see ([1],[2],[3],[5]). A proof of Theorem 2.2 (The binomial theorem) is given in [4]. Throughout this paper, $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{Z}_0^+$ denote the sets of integers, positive integers and positive integers with zero respectively.

**Definition 2.1.** Let $m \in \mathbb{Z}^+$, $r \in \mathbb{Z}_0^+$ and $a, b \in \mathbb{Z}$ such that $r \leq m - 1$.

1) We say that $a$ is congruent to $b$ modulo $m$ if $m$ divides $a - b$.

   We use the notation $a \equiv b \pmod{m}$ to indicate that $a$ is congruent to $b$ modulo $m$.

2) Define the set $\mathcal{N}(m, r) = \{ q \in \mathbb{Z} : q \equiv r \pmod{m} \}$.

   **Note that:** since $\mathcal{N}(m, r)$ is an infinitely countable set, we can label elements in $\mathcal{N}(m, r)$ as follows: ..., $q_{i-1}, q_i, q_{i+1}, ...$ where $q_i = im + r$. One says that $q_i$ and $q_{i+1}$ are consecutive congruent numbers in $\mathcal{N}(m, r)$.

**Theorem 2.2.** (The binomial theorem) Let $x$ and $y$ be variables and let $n$ be a positive integer. Then $(x + y)^n = \sum_{k=0}^{n} C_k^n x^{n-k} y^k$, where $C_k^n = \frac{n!}{k!(n-k)!}$.

**Proof.** See the proof in [4].
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**Lemma 2.3.** Let $m \in \mathbb{Z}^+$ and $r \in \mathbb{Z}_0^+$ such that $r \leq m - 1$. If $q_i$ and $q_{i+1}$ are consecutive congruent numbers in $\mathcal{N}(m, r)$ then $q_i < q_{i+1}$ and $q_{i+j} = q_i + mj$, for all $j \in \mathbb{Z}$.

**Proof.** The results follow immediately from the fact that $q_i$ and $q_{i+1}$ are consecutive congruent numbers in $\mathcal{N}(m, r)$.

\[ \square \]

## 3 Main Results

In this section, we prove the main Theorem 3.4 for recursive summation of the $n$th powers of consecutive congruent numbers. In Corollary 3.5 we use Theorem 3.4 to obtain explicit summation formulas for the first, second and third powers of consecutive congruent numbers.

We begin by stating and proving some lemmas.

**Lemma 3.1.** Let $m \in \mathbb{Z}^+$ and $r \in \mathbb{Z}_0^+$ such that $r \leq m - 1$.

If $q_i \in \mathcal{N}(m, r)$ then $\sum_{j=0}^{n} (q_{i+j}^n - q_{i+j-1}^n) = \sum_{k=1}^{n} (-1)^{k+1} C_k^n q_{i+j}^{n-k} m^k$, whenever $n \in \mathbb{Z}^+$ and $j \in \mathbb{Z}$.

**Proof.** According to the binomial theorem and $q_i \in \mathcal{N}(m, r)$, we have

\[
q_{i+j}^n - q_{i+j-1}^n = q_i^n - (q_i^j - m)^n \\
= q_i^n + \sum_{k=0}^{n} (-1)^{k+1} C_k^n q_i^{n-k} m^k \\
= \sum_{k=1}^{n} (-1)^{k+1} C_k^n q_i^{n-k} m^k, \quad \text{whenever } n \in \mathbb{Z}^+ \text{ and } j \in \mathbb{Z}.
\]

\[ \square \]

**Lemma 3.2.** Let $m \in \mathbb{Z}^+$ and $r \in \mathbb{Z}_0^+$ such that $r \leq m - 1$.

If $q_i \in \mathcal{N}(m, r)$ then $\sum_{j=0}^{s} (q_{i+j}^n - q_{i+j-1}^n) = q_i^n - q_{i-1}^n$, where $s \in \mathbb{Z}_0^+$ and $n \in \mathbb{Z}^+$.

**Proof.** By expanding $\sum_{j=0}^{s} (q_{i+j}^n - q_{i+j-1}^n)$, we obtain

\[
\sum_{j=0}^{s} (q_{i+j}^n - q_{i+j-1}^n) = (q_i^n - q_{i-1}^n) + (q_{i+1}^n - q_i^n) + \cdots + (q_{i+s}^n - q_{i+s-1}^n) \\
= q_i^n - q_{i-1}^n, \quad \text{whenever } s \in \mathbb{Z}_0^+ \text{ and } n \in \mathbb{Z}^+.
\]

\[ \square \]
Lemma 3.3. Let \( m \in \mathbb{Z}^+ \) and \( r \in \mathbb{Z}_0^+ \) such that \( r \leq m - 1 \).
If \( q_i \in \mathcal{N}(m, r) \) then

\[
\sum_{j=0}^{s} \left( \sum_{k=1}^{n} (-1)^{k+1} C_k^n q_{i+j}^{n-k} m^k \right) = nm \sum_{j=0}^{s} q_{i+j}^{n-1} + \sum_{j=0}^{s} \left( \sum_{k=2}^{n} (-1)^{k+1} C_k^n q_{i+j}^{n-k} m^k \right),
\]

whenever \( s \in \mathbb{Z}_0^+ \) and \( n \in \mathbb{Z}^+ \) with \( n \geq 2 \).

Proof. The result is obtained by expanding \( \sum_{j=0}^{s} \left( \sum_{k=1}^{n} (-1)^{k+1} C_k^n q_{i+j}^{n-k} m^k \right) \). We find

\[
\sum_{j=0}^{s} \left( \sum_{k=1}^{n} (-1)^{k+1} C_k^n q_{i+j}^{n-k} m^k \right) = \sum_{j=0}^{s} \left( C_1^n q_{i+j}^{n-1} m + \sum_{k=2}^{n} (-1)^{k+1} C_k^n q_{i+j}^{n-k} m^k \right) = nm \sum_{j=0}^{s} q_{i+j}^{n-1} + \sum_{j=0}^{s} \left( \sum_{k=2}^{n} (-1)^{k+1} C_k^n q_{i+j}^{n-k} m^k \right),
\]

whenever \( s \in \mathbb{Z}_0^+ \) and \( n \in \mathbb{Z}^+ \) with \( n \geq 2 \).

\[\square\]

Theorem 3.4. Let \( m \in \mathbb{Z}^+ \) and \( r \in \mathbb{Z}_0^+ \) such that \( r \leq m - 1 \).
If \( q_i \in \mathcal{N}(m, r) \) then

\[
m(n + 1) \sum_{j=0}^{s} q_{i+j}^{n} = q_{i+s}^{n+1} - q_{i-1}^{n+1} + \sum_{k=2}^{n+1} \left( (-1)^k C_k^{n+1} m^k \sum_{j=0}^{s} q_{i+j}^{n+1-k} \right),
\]

whenever \( s \in \mathbb{Z}_0^+ \) and \( n \in \mathbb{Z}^+ \).

Proof. For \( s \in \mathbb{Z}_0^+ \), \( n \in \mathbb{Z}^+ \) and \( q_i \in \mathcal{N}(m, r) \), we can apply the result in Lemma 3.3 and obtain

\[
m(n + 1) \sum_{j=0}^{s} q_{i+j}^{n} = \sum_{j=0}^{s} \left( \sum_{k=1}^{n+1} (-1)^{k+1} C_k^{n+1} q_{i+j}^{n+1-k} m^k \right) = \sum_{j=0}^{s} \left( \sum_{k=1}^{n+1} (-1)^{k+1} C_k^{n+1} q_{i+j}^{n+1-k} m^k \right).
\]

(1)

If we use Lemma 3.1 the left hand side term of (1) can be simplified as follows:

\[
m(n + 1) \sum_{j=0}^{s} q_{i+j}^{n} + \sum_{j=0}^{s} \left( \sum_{k=2}^{n+1} (-1)^{k+1} C_k^{n+1} q_{i+j}^{n+1-k} m^k \right) = \sum_{j=0}^{s} (q_{i+j}^{n+1} - q_{i+j-1}^{n+1}).
\]

(2)
Again, by applying Lemma 3.2 to the right hand side term of (2), we obtain,
\[ m(n + 1) \sum_{j=0}^{s} q_{i+j}^n + \sum_{j=0}^{s} \left( \sum_{k=2}^{n+1} (-1)^{k+1} C_k^{n+1} q_{i+j}^{n+1-k} m^k \right) = q_{i+s}^{n+1} - q_{i-1}^{n+1}, \]
so that
\[ m(n + 1) \sum_{j=0}^{s} q_{i+j}^n = q_{i+s}^{n+1} - q_{i-1}^{n+1} + \sum_{k=2}^{n+1} (-1)^{k} C_k^{n+1} m^k \sum_{j=0}^{s} q_{i+j}^{n+1-k} \]

**Corollary 3.5.** Let \( m \in \mathbb{Z}^+ \) and \( r \in \mathbb{Z}_0^+ \) such that \( r \leq m - 1 \). If \( q_i \in \mathcal{N}(m, r) \) then for each \( s \in \mathbb{Z}_0^+ \), the following formulas hold:

1) \[ 2m \sum_{j=0}^{s} q_{i+j} = q_{i+s}^2 - q_{i-1}^2 + m^2(s + 1) \text{ and more precisely,} \]
\[ \sum_{j=0}^{s} q_{i+j} = \frac{(s + 1)}{2} [m(2i + s) + 2r]. \]

2) \[ 3m \sum_{j=0}^{s} q_{i+j}^2 = q_{i+s}^3 - q_{i-1}^3 + \frac{m^2(s + 1)}{2} [m(6i + 3s - 2) + 6r]. \]

3) \[ 4m \sum_{j=0}^{s} q_{i+j}^3 = q_{i+s}^4 - q_{i-1}^4 + 2m(q_{i+s}^3 - q_{i-1}^3) + m^3(s + 1)[m(2i + s - 1) + 2r]. \]

**Proof.** Proof of 1). Using the results of Theorem 3.4, we deduce,
\[ 2m \sum_{j=0}^{s} q_{i+j} = q_{i+s}^2 - q_{i-1}^2 + m^2 \sum_{j=0}^{s} 1 = q_{i+s}^2 - q_{i-1}^2 + m^2(s + 1). \]

Then, we can write
\[ 2m \sum_{j=0}^{s} q_{i+j} = q_{i+s}^2 - q_{i-1}^2 + m^2(s + 1) \]
\[ = (q_{i+s} - q_{i-1})(q_{i+s} - q_{i-1}) + m^2(s + 1) \]
\[ = [m(s + 1)][m(2i + s - 1) + 2r] + m^2(s + 1) \]
\[ = m(s + 1)[m(2i + s) + 2r]. \]

Thus, finally
\[ \sum_{j=0}^{s} q_{i+j} = \frac{(s + 1)}{2} [m(2i + s) + 2r]. \]
Proof of 2). We use the results of Theorem 3.4 and Equation (3). Using Equation (3), we obtain,

\[
\sum_{k=2}^{3} \left( (-1)^k C_k^3 m^k \sum_{j=0}^{s} q_{i+j}^{3-k} \right) = 3m^2 \sum_{j=0}^{s} q_{i+j} - m^3 \sum_{j=0}^{s} 1 \\
= 3m^2 \frac{(s+1)}{2} [m(2i + s) + 2r] - m^3(s + 1) \\
= \frac{m^2(s + 1)}{2} [m(6i + 3s - 2) + 6r].
\]

(4)

Finally, using the results of Theorem 3.4 and Equation (4), we deduce

\[
3m \sum_{j=0}^{s} q_{i+j}^2 = q_{i+s}^3 - q_{i-1}^3 + \frac{m^2(s + 1)}{2} [m(6i + 3s - 2) + 6r].
\]

(5)

Proof of 3). The proof uses the results of Theorem 3.4 and Equations (3) and (5).

From the results of (3) and (5), we obtain

\[
\sum_{k=2}^{4} \left( (-1)^k C_k^4 m^k \sum_{j=0}^{s} q_{i+j}^{4-k} \right) \\
= 6m^2 \sum_{j=0}^{s} q_{i+j}^2 - 4m^3 \sum_{j=0}^{s} q_{i+j} + m^4 \sum_{j=0}^{s} 1 \\
= 2m \left( q_{i+s}^3 - q_{i-1}^3 + \frac{m^2(s + 1)}{2} [m(6i + 3s - 2) + 6r] \right) \\
- 2m^3 ((s + 1)[m(2i + s) + 2r]) + m^4(s + 1) \\
= 2m \left( q_{i+s}^3 - q_{i-1}^3 \right) + m^3(s + 1) [m(2i + s - 1) + 2r].
\]

(6)

Then using Theorem 3.4 and Equation (6), we obtain

\[
4m \sum_{j=0}^{s} q_{i+j}^3 = q_{i+s}^4 - q_{i-1}^4 + 2m(q_{i+s}^3 - q_{i-1}^3) + m^3(s + 1)[m(2i + s - 1) + 2r].
\]

\[
\square
\]

4 Conclusions and Applications

For given positive integers \( m \) and \( r \) such that \( 0 \leq r < m \), we define the set \( \mathcal{N}(m, r) = \{ q \in \mathbb{Z} : q \equiv r \pmod{m} \} \). The elements in \( \mathcal{N}(m, r) \) can be labelled consecutively as \( q_i = im + r, \ i \in \mathbb{Z} \). The important conclusion of this study is
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that the summation of the \( n \)th powers of consecutive congruent numbers can be carried out using the recursive form

\[
m(n + 1) \sum_{j=0}^{s} q_{i+j}^n = q_{i+s}^{n+1} - q_{i-1}^{n+1} + \sum_{k=2}^{n+1} (-1)^k C_{n}^{k+1} m^k \sum_{j=0}^{s} q_{i+j-k}^{n+1-k},
\]

where \( s \in \mathbb{Z}_0^+, n \in \mathbb{Z}^+ \) and \( C_{n}^{k+1} = \frac{(n+1)!}{k!(n-k+1)!} \).

Using this recursive form, we obtained the following explicit summation formulas for the first, second and third powers of consecutive congruent numbers:

1) \[
\sum_{j=0}^{s} q_{i+j} = \frac{(s + 1)}{2}[m(2i + s) + 2r].
\]

2) \[
3m \sum_{j=0}^{s} q_{i+j}^2 = q_{i+s}^3 - q_{i-1}^3 + \frac{m^2(s + 1)}{2}[m(6i + 3s - 2) + 6r].
\]

3) \[
4m \sum_{j=0}^{s} q_{i+j}^3 = q_{i+s}^4 - q_{i-1}^4 + 2m(q_{i+s}^3 - q_{i-1}^3) + m^3(s + 1)[m(2i + s - 1) + 2r].
\]

The above results can be used to solve some practical problems. For example, to find the total surface (or volume or mass) of a set \( s \) objects in which

- all of them are cubes but of different sizes, and
- the difference in lengths of sides of the consecutive objects is equal to \( m \) units.

Those objects have been shown in the following figure.

In general, the calculation of sums of an \( n \)th power of consecutive congruent numbers can be a long and tedious task. However, the calculation can be carried out easily by using the recursive summation form of the \( n \)th powers of consecutive congruent numbers and the summation formulas for the first, second and third powers.

We now give two examples.
Example 4.1. Suppose a plastic factory wants to produce a package containing 50 different plastic boxes in which all boxes are cubes but of different sizes. When arranged in the package, the differences in length of sides of boxes that are adjacent or consecutive will be equal to 10 centimeters and the smallest box will be assumed to have a volume 27 cubic centimeters. The question is what will be the total volume of these 50 plastic boxes.

Solution. We can analyze and solve this problem as follows: Obviously, we see that \( n = 3, \ s = 50, \ m = 10, \ r = 3 \) and \( i = 0 \), then we obtain

\[
q_i = r, q_{i+s}^4 - q_i^4 = 64,013,551,680, \ 2m(q_{i+s}^3 - q_{i-1}^3) = 2,545,277,400,
\]

\[
m^3(s + 1) = 51,000 \text{ and } m(2i + s - 1) + 2r = 496.
\]

Therefore, we find

\[
4m \sum_{j=0}^{s} q_{i+j}^3 = q_{i+s}^4 - q_i^4 + 2m(q_{i+s}^3 - q_{i-1}^3) + m^3(s + 1)[m(2i + s - 1) + 2r]
\]

\[
= 66,558,880,576.
\]

Thus, \( \sum_{j=0}^{s} q_{i+j}^3 = 1,663,972,014 \) cubic centimeters and we conclude that the total volume of all plastic boxes is 1,663,972,014 cubic centimeters.

Example 4.2. Suppose one wants to design a shelf whose shape looks like a pyramid. It requires some conditions as follows: total surface area of all stages is 45,275 square centimeters, the area of a stage on the upper level is less than that on the lower level, numbers of levels are 10, each stage on any level is square, side differences of stages that are adjacent or consecutive are 10 centimeters. In addition, the area of the highest stage is 25 square centimeters. The question is whether it is possible to design this shelf in a pyramid form with the above conditions.

Solution. We can solve this problem as follows: Clearly, we see that \( n = 2, \ s = 10, \ m = 10, \ r = 5 \) and \( i = 0 \). Then we have

\[
q_i = r, q_{i+s}^3 - q_i^3 = 1,157,750,
\]

\[
m^2(s + 1) = 5,500 \text{ and } [m(6i + 3s - 2) + 6r] = 310.
\]

Hence, we infer the results,

\[
3m \sum_{j=0}^{s} q_{i+j}^2 = q_{i+s}^3 - q_i^3 + \frac{m^2(s + 1)}{2}[m(6i + 3s - 2) + 6r] = 2,862,750.
\]
Therefore, it implies that \( \sum_{j=0}^{s} q_{i+j}^2 = 95,425 \) square centimeters. Thus, the total area of all stages of the shelf would be 95,425 square centimeters and not 45,275 square centimeters as required. Hence the given conditions cannot be satisfied.

\[ \diamond \]

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References


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