Toeplitz Operators with $L^1$ Symbols on the Weighted Bergman Spaces

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Abstract

This paper characterizes the boundedness and compactness of a Toeplitz operator on the weighted Bergman space with a $L^1$ symbol. This result extends known results in the cases when the symbol is either a positive $L^1$ function, an $L^\infty$ function or a general $BMO^1$ function. In addition, we also give some estimates about the norm and essential norm of Toeplitz operators with $L^1$ symbols.

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1 Introduction

Throughout this paper, let $n \geq 2$ be a fixed integer. Denote the unit ball in $\mathbb{C}^n$ by $\mathbb{B}_n$, and let $dV$ be the normalized lebesgue volume measure on $\mathbb{B}_n$. For $-1 < \alpha < \infty$, we denote by $dV_\alpha$ the measure on $\mathbb{B}_n$ defined by

$$dV_\alpha(z) = C_\alpha(1 - |z|^2)^\alpha dV(z),$$

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where \( C_\alpha = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(\alpha+1)} \) is a normalizing constant such that \( V_\alpha(\mathbb{B}_n) = 1 \).

For \( 1 \leq p < \infty \), we write \( \| \cdot \|_p \) for the norm on \( L^p(\mathbb{B}_n, dV_\alpha) \) and \( \langle \cdot, \cdot \rangle \) for the inner product on \( L^2(\mathbb{B}_n, dV_\alpha) \). The weighted Bergman space \( A^2_\alpha \) is the space of analytic functions on \( \mathbb{B}_n \) that are also in \( L^2(\mathbb{B}_n, dV_\alpha) \). Reproducing kernels \( K_w \) and normalized reproducing kernels \( k_w \) in \( A^2_\alpha \) are given by, respectively,

\[
K_w(z) = \frac{1}{(1 - \overline{w}z)^{n+1+\alpha}} \quad \text{and} \quad k_w(z) = \frac{(1 - |w|^2)^{(n+1+\alpha)/2}}{(1 - z\overline{w})^{n+1+\alpha}}
\]

for \( z, w \in \mathbb{B}_n \). For every \( h \in A^2_\alpha \), we have \( \langle h, K_w \rangle = h(w) \) for all \( w \in \mathbb{B}_n \).

We let \( T_f \) denote the Toeplitz operator with symbol \( f \) on \( A^2_\alpha \) defined by

\[
T_f g = Pg,
\]

where \( P \) is the orthogonal projection from \( L^2(\mathbb{B}_n, dV_\alpha) \) onto \( A^2_\alpha \), that is,

\[
(Pg)(w) = \langle g, K_w \rangle = \int_{\mathbb{B}_n} \frac{g(z)}{(1 - wz)^{n+1+\alpha}} dV_\alpha(z)
\]

for \( g \in L^2(\mathbb{B}_n, dV_\alpha) \) and \( w \in \mathbb{B}_n \).

Since the Bergman projection function \( P \) can be extended to \( L^1(\mathbb{B}_n, dV_\alpha) \), the operator \( T_f \) is well defined on \( H^\infty(\mathbb{B}_n) \), the space of bounded analytic functions on \( \mathbb{B}_n \), which is dense in \( A^2_\alpha \). Hence, \( T_f \) is always densely defined on \( A^2_\alpha \) (See [1]).

In the case of Hardy space, bounded Toeplitz operators arise from bounded symbols and there are no nontrivial compact Toeplitz operators (See [2] and [3]). In the Bergman space setting, however, there are lots of nontrivial compact Toeplitz operators [4]. In fact, Stroethoff K. [4] first constructed a nonzero compact Toeplitz operator \( T_f \) such that \( f^2 = 1 \). Some unbounded symbols induced bounded Toeplitz operators and even compact Toeplitz operators. Miao J. and Zheng D. [5] showed that if on the Bergman space \( L^p_\alpha \) for \( p > 1 \), an operator equals a finite sum of finite products of Toeplitz operators with symbols in \( BT \), the operator is compact if and only if the Berezin transform of the operator vanishes on the boundary of the unit disk. Dieudonne A. [6] characterized the boundedness and compactness of Toeplitz operators with \( L^1(\mathbb{D}) \) symbols on the Bergman space.

The analogous results are true on the weighted Bergman space of several complex variables. In this paper, we characterize the boundedness and compactness of Toeplitz operators with \( L^1(\mathbb{B}_n, dV_\alpha) \) symbols. And some estimates about the norm and essential norm of Toeplitz operators with symbols in \( L^1(\mathbb{B}_n, dV_\alpha) \) are obtained.

Our result extends known results in the cases when the symbol is either a positive \( L^1 \) function, an \( L^\infty \) function or a general \( BMO^1 \) function earlier proved respectively by Zhu K. [7], Axler S. and Zheng D. [8] and Zorboska...
N. [9]. An attempt to prove this theorem was undertaken by Zorboska N. [9]. Our methods of proof are adapted from methods used in both in [8] and [9] which we combine with a result due to Luecking D. [10].

Throughout the paper, we will use the letter $C$ to denote a generic positive constant that can change its value at each occurrence.

## 2 Main results and Proofs

For $z, w \in \mathbb{B}_n$, let $\varphi_z$ be the analytic automorphism of the unit ball $\mathbb{B}_n$ defined by

$$
\varphi_z(w) = \frac{z - P_zw - sQ_zw}{1 - \langle w, z \rangle}
$$

where $P_z$ is the orthogonal projection of $\mathbb{C}^n$ into the subspace $[z]$ generated by $z$, that is, $P_0 = 0$ and if $z \neq 0$, $P_zw = \frac{\langle w, z \rangle}{\langle z, z \rangle}z$ and $Q_z = I - P_z$ is the projection onto the orthogonal complement of $[z]$, $s = \sqrt{1 - |z|^2}$.

For $z \in \mathbb{B}_n$, let $U_z : A^2_\alpha \rightarrow A^2_\alpha$ be the unitary operator defined by

$$
U_zf = (f \circ \varphi_z)k_z.
$$

Then $U_z$ is a self-adjoint unitary operator.

We shall use the following lemmas:

**Lemma 2.1.** For $f \in L^1(\mathbb{B}_n, dV_\alpha)$, if $T_f$ is bounded on $A^2_\alpha$, then

$$
U_zT_fU_z = T_{f \circ \varphi_z}.
$$

**Lemma 2.2.** Let $f \in L^1(\mathbb{B}_n, dV_\alpha)$ and $h \in A^2_\alpha$. Then

(a) $(T_fK_z)(u) = K_z(u)P_f(\varphi_z(u)(\varphi_z(u))$;

(b) $(T_fK_z)(u) = (T_fK_u)(z)$;

(c) $(T_fh)(v) = \int_{\mathbb{B}_n} h(u)(T_fK_u)(v)dV_\alpha(u)$;

(d) $T_f^*h = T_f^*h$ if $T_f$ is bounded on $A^2_\alpha$.

By [6], it is not difficult to get that

**Lemma 2.3.** Suppose $0 < r < \frac{1}{2}$ and $0 < q < p$. Let $\mu$ be a positive Borel measure on unit ball and $D(w, r)$ be the Bergman metric ball, define the function $k(w) = \frac{\mu(D(w, r))}{V_\alpha(D(w, r))}$, then the estimate

$$
\left( \int_{\mathbb{B}_n} |f|^q d\mu \right)^{1/q} \leq C \left( \int_{\mathbb{B}_n} |f|^p dV_\alpha \right)^{1/p}
$$

holds if and only if $k$ is in $L^s(\mathbb{B}_n, dV_\alpha)$, where $\frac{1}{s} + \frac{q}{p} = 1$. Moreover, $C = c\|k\|_s^{\frac{1}{q}}$. 

For $\epsilon > 0$, we define an operator $S$ by
\[
(Sf)(z) = \int_{\mathbb{B}_n} |f(v)| \cdot \frac{1}{(1 - |z|^2)^{(n+1+\alpha)/2}} \cdot \frac{1}{(1 - |v|^2)^{(n+1+\alpha)/2}} \cdot \frac{1}{(1 - |\varphi_z(v)|^2)^{(2\epsilon - 1)(n+1+\alpha)/2}} dV_{\alpha}(v).
\]

**Lemma 2.4.** Let $p > 1$ and $0 < \epsilon < \frac{1}{(n+1+\alpha)p'}$ where $p'$ is the conjugate exponent of $p$. Then there exists a constant $C = C(p, \epsilon)$ such that for all $f \in L^1(\mathbb{B}_n)$ the following estimate holds:
\[
|Sf(z)| \leq C(K_z(z))^{\epsilon} \|f\|_p.
\]

**Remark** For $\epsilon > 0$, let $g(v) = (K_v(v))^\epsilon$. Then it is easy to get that for all $h \in L^1(\mathbb{B}_n, dV_{\alpha})$ and all $u \in \mathbb{B}_n$, the following identity is true:
\[
\int_{\mathbb{B}_n} |K_u(v)P(h \circ \varphi_u)(\varphi_u(v))|g(v)dV_{\alpha}(v) = (S(P(h \circ \varphi_u)))(u).
\]

Now let’s see the boundedness of Toeplitz operators with $L^1(\mathbb{B}_n, dV_{\alpha})$ symbols.

**Theorem 2.5.** Let $f \in L^1(\mathbb{B}_n, dV_{\alpha})$, then $T_f$ extends to a bounded operator on $A^2_{\alpha}$ if and only if
\[
\sup_{z \in \mathbb{B}_n} \|T_f \circ \varphi_z\|_2 < \infty \quad \text{and} \quad \sup_{z \in \mathbb{B}_n} \|T_f \bar{\circ} \varphi_z\|_2 < \infty.
\]

**Proof.** Let $f \in L^1(\mathbb{B}_n, dV_{\alpha})$, $h \in A^2_{\alpha}$ and $v \in \mathbb{B}_n$. Then by lemma 2.2(c) we have that
\[
(T_fh)(v) = \int_{\mathbb{B}_n} h(u)(T_fK_u)(v)dV_{\alpha}(u),
\]
which implies that $T_f$ is an integral operator with kernel $(T_fK_u)(v)$. By Schur’s lemma [11] and [7], if there exist a positive measurable function $g$ on $\mathbb{B}_n$ and constants $c_1$, $c_2$ such that
\[
\int_{\mathbb{B}_n} |(T_fK_u)(v)|g(v)dV_{\alpha}(v) \leq c_1g(u) \quad (1)
\]
for all $u \in \mathbb{B}_n$ and
\[
\int_{\mathbb{B}_n} |(T_fK_u)(v)|g(u)dV_{\alpha}(u) \leq c_2g(v) \quad (2)
\]
for all $v \in \mathbb{B}_n$, then $T_f$ is bounded on $A^2_{\alpha}$ and $\|T_f\| \leq \sqrt{c_1c_2}$.

For $\epsilon > 0$, take $g(v) = (K_v(v))^\epsilon$. Then lemma 2.2(a) implies that the left hand side of (1) is equal to

$$\int_{\mathbb{B}_n} |K_u(v)P(f \circ \varphi_u)(\varphi_u(v))|g(v)dV_\alpha(v). \quad (3)$$

According to (3) and Remark, the left hand side of (1) can be written as

$$(S(P(f \circ \varphi_u)))(u).$$

Now for $0 < \epsilon < \frac{1}{2(n+1+\alpha)}$ we obtain from lemma 2.4 that there exists a constant $C > 0$ such that

$$(S(P(f \circ \varphi_u)))(u) \leq C\|P(f \circ \varphi_u)\|_2(K_u(u))^\epsilon$$

$$\leq C \sup_{u \in \mathbb{B}_n} \|P(f \circ \varphi_u)\|_2(K_u(u))^\epsilon.$$

This gives estimate (1) with $c_1 = C \sup_{u \in \mathbb{B}_n} \|P(f \circ \varphi_u)\|_2$.

Next let us prove the estimate in (2). We use lemma 2.2(b) to get that the left hand side of (2) is at most

$$\int_{\mathbb{B}_n} |(T_fK_v)(u)|g(u)dV_\alpha(u) = \int_{\mathbb{B}_n} |K_v(u)P(\bar{f} \circ \varphi_v)(\varphi_v(u))|g(u)dV_\alpha(u)$$

$$= S(P(\bar{f} \circ \varphi_v))(v).$$

If $0 < \epsilon < \frac{1}{2(n+1+\alpha)}$, then lemma 2.4 implies that there exists a constant $C > 0$ such that

$$SP(\bar{f} \circ \varphi_v)(v) \leq C \sup_{u \in \mathbb{B}_n} \|P(\bar{f} \circ \varphi_v)\|_2g(v).$$

This gives estimate (2) with $c_2 = C \sup_{v \in \mathbb{B}_n} \|P(\bar{f} \circ \varphi_v)\|_2$. Thus by Schur’s Lemma $T_f$ is bounded on $A^2_{\alpha}$.

Conversely, if $T_f$ is bounded, then using Lemma 2.1 we have

$$\sup_{z \in \mathbb{B}_n} \|T_{f \circ \varphi_z}1\|_2 = \sup_{z \in \mathbb{B}_n} \|U_zT_fU_z1\|_2 = \sup_{z \in \mathbb{B}_n} \|T_fU_z1\|_2 \leq \|T_f\|_2 < \infty.$$

Since $T_f$ is bounded implies $T_f$ is bounded, this also shows in a similar manner that $\sup_{z \in \mathbb{B}_n} \|T_{f \circ \varphi_z}1\|_2$ is finite.

We present some necessary results that will be useful in our proof of the compactness of $T_f$, for $f \in L^1(\mathbb{B}_n,dV_\alpha)$.

**Lemma 2.6.** Let $A$ be a bounded operator on $A^2_{\alpha}$ and let $U_z$ be the unitary operator on $A^2_{\alpha}$. Then the property $A \rightarrow 0$ as $z \rightarrow \partial \mathbb{B}_n$ implies that $U_zAU_z \rightarrow 0$ weakly in $A^2_{\alpha}$ as $z \rightarrow \partial \mathbb{B}_n$. 
Lemma 2.7. Let \( f \in L^1(\mathbb{B}_n, dV_\alpha) \), suppose that \( T_f \) is bounded on \( A^2_\alpha \) and \( \tilde{f}(z) \rightarrow 0 \) as \( z \rightarrow \partial \mathbb{B}_n \). Then \( \|T_{f^{op_q}}\| \rightarrow 0 \) as \( z \rightarrow \partial \mathbb{B}_n \) for \( 1 \leq q \leq 2 \).

Theorem 2.8. Let \( f \in L^1(\mathbb{B}_n, dV_\alpha) \), suppose that \( T_f \) is bounded on \( A^2_\alpha \) and \( \tilde{f}(z) \rightarrow 0 \) as \( z \rightarrow \partial \mathbb{B}_n \). Then \( T_f \) is a compact operator on \( A^2_\alpha \).

Proof. Let \( h \in A^2_\alpha \). Then for \( f \in L^1(\mathbb{B}_n, dV_\alpha) \) such that \( T_f \) is bounded on \( A^2_\alpha \), we get:

\[
(T_f h)(v) = <T_f h, K_v> = <h, T_f^* K_v>.
\]

But according to Lemma 2.2(c) and (d), we have:

\[
(T_f^* K_v)(u) = \overline{T_f K_u(v)}.
\]

Substituting (5) in (4) gives:

\[
(T_f h)(v) = \int_{\mathbb{B}_n} h(u)\overline{T_f K_u(v)}dV_\alpha(u) = \int_{\mathbb{B}_n} h(u)(T_f K_u(v))dV_\alpha(u).
\]

For \( r \in (0, 1) \), define an operator \( T_f^{[r]} \) on \( A^2_\alpha \) by

\[
(T_f^{[r]} h)(v) = \int_{r\mathbb{B}_n} h(u)(T_f K_u(v))dV_\alpha(u).
\]

Then \( T_f^{[r]} \) is an integral operator with kernel \( (T_f K_u(v))\chi_{r\mathbb{B}_n}(u) \). This operator is a Hilbert Schmidt operator, since

\[
\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} |(T_f K_u(v))\chi_{r\mathbb{B}_n}(u)|^2dV_\alpha(v)dV_\alpha(u) = \int_{r\mathbb{B}_n} \int_{\mathbb{B}_n} |(T_f K_u(v))|^2dV_\alpha(v)dV_\alpha(u)
\]

\[
= \int_{r\mathbb{B}_n} \|T_f K_u\|^2dV_\alpha(v)
\]

\[
\leq \|T_f\|^2 \int_{r\mathbb{B}_n} \frac{1}{(1 - |u|^2)^{n+1+\alpha}}dV_\alpha(u),
\]

which is finite since \( T_f \) is bounded on \( A^2_\alpha \) and the last integral is over a compact set. This proves that \( T_f^{[r]} \) is a compact operator on \( A^2_\alpha \).

Furthermore, we only need to show that

\[
\|T_f - T_f^{[r]}\|^2 \rightarrow 0 \text{ as } r \rightarrow 1^{-}
\]

where \( \|T_f - T_f^{[r]}\| \) denotes the norm operator of \( T_f - T_f^{[r]} \) as an operator from \( A^2_\alpha \) to itself. Now, for \( h \in A^2_\alpha \) we have

\[
(T_f - T_f^{[r]}h)(v) = \int_{\mathbb{B}_n} \chi_{\mathbb{B}_n \setminus r\mathbb{B}_n}(u)h(u)(T_f K_u(v))dV_\alpha(u).
\]
This implies $T_f - T_f^{[r]}$ extends to an integral operator on $L^2(\mathbb{B}_n, dV_\alpha)$ with kernel

$$(T_f K_u)(v)\chi_{\mathbb{B}_n \setminus r\mathbb{B}_n}(u).$$

By Schur’s lemma, if there exist a positive measurable function $g$ on $\mathbb{B}_n$ and constants $c_1, c_2$ such that

$$\int_{\mathbb{B}_n} |(T_f K_u)(v)\chi_{\mathbb{B}_n \setminus r\mathbb{B}_n}(u)|g(v)dV_\alpha(v) \leq c_1g(u) \quad (6)$$

for all $u \in \mathbb{B}_n$ and

$$\int_{\mathbb{B}_n} |(T_f K_u)(v)\chi_{\mathbb{B}_n \setminus r\mathbb{B}_n}(u)|g(u)dV_\alpha(u) \leq c_2g(v) \quad (7)$$

for all $v \in \mathbb{B}_n$, then

$$\|T_f - T_f^{[r]}\|^2 \leq c_1c_2.$$

For $\epsilon > 0$, take $g(v) = (K_v(v))^\epsilon$. Then using the same argument as in the proof of Theorem 2.5 we see that the left hand side of (6) can be written as

$$\chi_{\mathbb{B}_n \setminus r\mathbb{B}_n}(u)(S(P(h \circ \varphi_u)))(u).$$

Now for $1 \leq q < 2$ and for $0 < \epsilon < \frac{1}{(n+1+\alpha)q'}$ where $q'$ is the conjugate exponent of $q$. We obtain from Lemma 2.4 that there exists a constant $C > 0$ such that

$$\chi_{\mathbb{B}_n \setminus r\mathbb{B}_n}(u)(S(P(f \circ \varphi_u)))(u) \leq C\chi_{\mathbb{B}_n \setminus r\mathbb{B}_n}(u)\|P(f \circ \varphi_u)\|_q(K_u(u))^\epsilon \leq C\sup_{|u|>r}\|P(f \circ \varphi_u)\|_q(K_u(u))^\epsilon.$$

This gives estimate (6) with $c_1 = C\sup_{|u|>r}\|P(f \circ \varphi_u)\|_q$.

Similarly we get the left side of (7) is at most

$$C\sup_{v \in \mathbb{B}_n}\|P(\tilde{f} \circ \varphi_v)\|_2g(v).$$

This gives estimate (7) with $c_2 = C\sup_{v \in \mathbb{B}_n}\|P(\tilde{f} \circ \varphi_v)\|_2$

Finally, since $\min\{\frac{1}{(n+1+\alpha)q'}, \frac{1}{4}\} = \frac{1}{(n+1+\alpha)q'}$, the estimates (6) and (7) both hold for $g(v) = (K_v(v))^\epsilon$ with $\epsilon < \frac{1}{(n+1+\alpha)q'}$. By Schur’s lemma, we therefore get:

$$\|T_f - T_f^{[r]}\|^2 \leq c_1c_2,$$

where

$$c_1 = C\sup_{|u|>r}\|P(f \circ \varphi_u)\|_q \text{ and } c_2 = C'\sup_{v \in \mathbb{B}_n}\|P(\tilde{f} \circ \varphi_v)\|_2.$$

Since by Lemma 2.3, $c_1 \to 0$ as $r \to 1$ and $c_2 < \infty$, the conclusion

$$\|T_f - T_f^{[r]}\|^2 \to 0 \text{ as } r \to 1^-$$
follows.

Now we consider the norm and essential norm of the Toeplitz operator $T_f$ on $A^2_\alpha$ for $f \in L^1(\mathbb{B}_n, dV_\alpha)$. In [12], Englis showed that neither
\[
\|T_f\| \leq C \lim_{z \to \partial \mathbb{B}_n} \sup |\tilde{f}(z)| \quad \forall f \in L^\infty(\mathbb{D}, dA)
\]
nor
\[
\|T_f\| \leq C \sup_{z \in \mathbb{D}} |\tilde{f}(z)| \quad \forall f \in L^\infty(\mathbb{D}, dA)
\]
can hold for any constant $C$. Here $\|T_f\|_e$ denotes the essential norm of the Toeplitz operator $T_f$ defined by
\[
\|T_f\|_e = \inf_{K \in \mathcal{K}} \|T_f - K\|,
\]
where $\mathcal{K}$ is the set of compact operators on $A^2_\alpha$. Later, Nazarov told us that the inequality
\[
\|T_f\| \leq C \sup_{z \in \mathbb{D}} \|T_f k_z\|_2 \quad \forall f \in L^\infty(\mathbb{D}, dA)
\]
cannot hold for any constant $C$.

**Theorem 2.9.** For all $f \in L^1(\mathbb{B}_n, dV_\alpha)$ with $\|f\| \leq 1$, there is a constant $C$ such that
\[
\|T_f\| \leq C \left( \sup_{z \in \mathbb{B}_n} \|T_f k_z\|_2 \sup_{z \in \mathbb{B}_n} \|T_f k_z\|_2 \right)^{1/2},
\]
and
\[
\|T_f\|_e \leq C \left( \lim_{z \to \partial \mathbb{B}_n} \sup_{z \in \mathbb{B}_n} \|T_f k_z\|_2 \lim_{z \to \partial \mathbb{B}_n} \sup_{z \in \mathbb{B}_n} \|T_f k_z\|_2 \right)^{1/2}.
\]

**Proof.** For $h \in A^2_\alpha$ and $v \in \mathbb{B}_n$, we have
\[
(T_f h)(v) = \int_{\mathbb{B}_n} h(u)(T_f K_u)(v) dV_\alpha(u).
\]
Then Theorem 2.5 gives
\[
\|T_f\| \leq C \left( \sup_{u \in \mathbb{B}_n} \|P(f \circ \varphi_u)\|_2 \sup_{v \in \mathbb{B}_n} \|P(\tilde{f} \circ \varphi_v)\|_2 \right)^{1/2}.
\]

For $0 < r < 1$ and $0 < s < 1$, define an operator $K_{[r]}$ on $A^2_\alpha$ by
\[
(K_{[r]} h)(v) = \int_{r \mathbb{B}_n} h(u)(T_f K_u)(v) dV_\alpha(u),
\]
and an operator $K_{[r],[s]}$ on $A^2_\alpha$ by
\[
(K_{[r],[s]} h)(v) = \chi_{\mathbb{B}_n \setminus r \mathbb{B}_n} (v) \int_{\mathbb{B}_n \setminus r \mathbb{B}_n} h(u)(T_f K_u)(v) dV_\alpha(u).
\]
As in the proof of Theorem 2.5, both $K_{[r]}$ and $K_{[r],[s]}$ can be shown to be compact on $A^2_\alpha$.

If $r, s \in (0, 1)$, then $T_f - K_{[r]} - K_{[r],[s]}$ is the integral operator with kernel

$$(T_f K_u)(v)\chi_{\mathbb{B}_n \setminus B_n}(u)\chi_{\mathbb{B}_n \setminus B_n}(v).$$

The proof of Theorem 2.5 indicates that

$$\|T_f - K_{[r]} - K_{[r],[s]}\| \leq C \sqrt{c_1 c_2},$$

where $c_1 = \sup_{r \leq |u| < 1} \|P(f \circ \varphi_u)\|_2$, $c_2 = \sup_{s \leq |v| < 1} \|P(\bar{f} \circ \varphi_v)\|_2$. We have shown

$$\|T_f\|_e \leq C \left( \lim_{u \to \partial \mathbb{B}_n} \sup \|P(f \circ \varphi_u)\|_2 \lim_{v \to \partial \mathbb{B}_n} \sup \|P(\bar{f} \circ \varphi_v)\|_2 \right)^{1/2}.$$ 

References


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