Bounded Sequence of Symbol Classes of Pseudodifferential Operator

Shawgy Hussein

Sudan University of Science and Technology, Sudan
Shawgy2020@gmail.com

Mohammed El-Moudier Gobar

Al-Baha University, Kingdom of Saudi Arabia
mudeer.all@gmail.com

Abstract

We investigate the properties of an exotic sequence of symbol classes of pseudodifferential operators. A sequence of Sjöstrand's classes will be discuss in methods of time-frequency analysis. The time frequency leads to simple Sjöstrand's fundamental results of modifying generalizations.

Mathematics Subject Classification: 35S05, 47G30

Keywords: Pseudodifferential operators, exotic sequence of symbol classes, Wigner distribution, Gabor frame, short-time Fourier transform, spectral invariance, almost diagonalization, modulation space, Wiener's lemma

1. Introduction

In 1990/95 Sjöstrand's introduced a symbol class for pseudodifferential operators that contains the Hormander class $S^0_{0,0}$ and also includes non-smooth symbols. He proved three fundamental results about the $L^2$-boundedness the algebra property, and the Wiener's property. This work had considerable impact
on subsequent work in both hard analysis [9,10,28,42-44] and time-frequency analysis [11,23,24].

To make the connection of Sjöstrand's definition to time-frequency analysis:

Let \( g_j \in S(\mathbb{R}^d), j = 1,2, \ldots \) be sequence of a \( C^\infty \)-functions with the compact support satisfying the property \( \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^{2d}} g_j(t - k) = 1, \forall t \in \mathbb{R}^d \). Then a sequence of symbols \( \sigma_j \in S'(\mathbb{R}^d), j = 1,2, \ldots \) belongs to \( M^{\infty,1} \) the Sjöstrand's class if

\[
\sum_{j=1}^{\infty} \int_{\mathbb{R}^d} \sup_{z \in \mathbb{R}^d} |(\sigma_j M_j T_z g_j)| d\zeta < \infty
\]

The sequence of Weyl transforms of symbol \( \sigma_j(z, \zeta), j = 1,2, \ldots \) is defined as

\[
\sum_{j=1}^{\infty} \sigma_j^w f(x) = \sum_{j=1}^{\infty} \int_{\mathbb{R}^d} \sigma_j \left( \frac{x+y}{2}, \zeta \right) e^{2\pi i (x-y) \cdot \zeta} f(y) dy d\zeta
\]

Sjöstrand's proved the following fundamental results about the Weyl transform of symbols \( \sigma_j \in M^{\infty,1}(\mathbb{R}^{2d}), j = 1,2, \ldots \) [38,39].

(a) If \( \sigma_j \in M^{\infty,1}(\mathbb{R}^{2d}), j = 1,2, \ldots \), then \( \sigma_j^w \) are a bounded operators on \( L^2(\mathbb{R}^d) \).

(b) If \( \sigma_1, \sigma_2 \in M^{\infty,1}(\mathbb{R}^{2d}) \) and \( \tau^w = \sigma_1^w \sigma_2^w \), then \( \tau \in M^{\infty,1}(\mathbb{R}^{2d}) \); thus \( M^{\infty,1} \) is a (Banach) algebra of pseudodifferential operators.

(c) If \( \sigma_j \in M^{\infty,1}(\mathbb{R}^{2d}), j = 1,2, \ldots \) and \( \sigma_j^w \) is invertible on \( L^2(\mathbb{R}^d) \), then \( (\sigma_j^w)^{-1} = \tau^w \) for some \( \tau \in M^{\infty,1}(\mathbb{R}^{2d}) \). This is the Wiener property of \( M^{\infty,1} \).

For the classical symbol classes results of this type (see Beals [3]). The original proofs of Sjöstrand's were carried out in the realm of classical "hard" analysis. This line of investigation was deepened and extended in subsequent work by Boulkhemair, Héralds, and Toft [9,10,28,42-44].

Later it was discovered that Sjöstrand's class \( M^{\infty,1} \) is special case of a so-called modulation spaces. The family of modulation spaces had been studied in time-frequency analysis since the 1980s and later was also used in the theory of pseudodifferential operators. The action of pseudodifferential operators with classical symbols on modulation spaces was investigated by Tachizawa [41] in 1994: general modulation spaces as symbol classes for pseudodifferential operators were introduced in [23] independently of Sjöstrand's work. This line of investigation and the emphasis on time-frequency techniques was continued in [11,12,23,31,32].

To make the connection to time-frequency analysis, we introduce the operators of translation and modulation,
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\[ T_x f(t) = f(t - x) \]
\[ M_w f(t) = e^{2\pi i w \cdot t}, t, x, w \in \mathbb{R}^d \]

we note that

\[ \left( \sum_{j=1}^{\infty} \sigma_j, g_j (\cdot - z) \right) (\zeta) = \int_{\mathbb{R}^{2d}} \sum_{j=1}^{\infty} \sigma_j(t) \tilde{g}_j(t - z) e^{-2\pi i \zeta \cdot t} dt = \sum_{j=1}^{\infty} \langle \sigma_j, M_{\zeta} T_z g_j \rangle \]

This is the so-called a series of short-time Fourier transforms. In view of (3) a distribution belongs to Sjöstrand's class, if its short-time Fourier transform satisfies the condition \( \sum_{j=1}^{\infty} \int_{\mathbb{R}^{2d}} \sup_{z \in \mathbb{R}^{2d}} |\sigma_j, M_{\zeta} T_z g_j| d\zeta < \infty. \)

More generally the modulation spaces are defined by imposing a weighted \( L^p - norm \) on the series short-time Fourier transform. This class of function spaces was introduced by H.G. Feichtinger in 1983 [15] and [14,16] and has been studied extensively. The modulation spaces have turned out to be the appropriate function and distribution spaces for many problems in time-frequency analysis.

The objective of this paper shown by [24] is to give the natural proofs of Sjöstrand's results. The definition of Sjöstrand's by means of short-time Fourier transform (3) suggests that the mathematics of translation and modulation operators, in other words, time-frequency analysis, should enter in the proofs. Although "natural" is a debatable notion in mathematics, we argue that methods of time-frequency analysis should simply the original proofs and shed new light on Sjöstrand's results. Currently, several different proofs exist for the boundedness and the algebra property, both in the context of "hard analysis" and of time-frequency analysis. However, for the Wiener property only Sjöstrand's original "hard analysis" proof was known, and it was an open problem to find an alternative proof. Karlheing Gröchenig gives conceptually new and technically simple proofs of Sjöstrand's fundamental results[24], but we present a modification of the boundedness property of the operator of the Weyl transform with respect to a sequence of symbols on Hilbert space. We give a result expressing the sequence Wigner distributions of a time-frequency shifts as a limited time-frequency shifts.

Time-Frequency methods provide detailed information on which class of function spaces Weyl transforms with symbols in \( M^{\infty,1} \) act boundedly.

The time-frequency methods suggest the appropriate and maximal generalization of Sjöstrand's results (to weighted modulation spaces).

Although we restricts our attention to Weyl transforms and modulation spaces on \( \mathbb{R}^d \), all concepts can be defined on arbitrary locally compact abelian groups.
One may conjecture that Sjöstrand's results hold for (pseudodifferential) operators on $L^2$ of locally compact abelian groups as well. In that case time-frequency methods hold more promise than analysis methods. We show that Weyl transforms with symbols in Sjöstrand's class are almost diagonalized by Gabor frames. This may not be surprising, because it is well-known that pseudodifferential operators with classical symbols are almost diagonalized by wavelet bases and local Fourier based [33,35]. What is remarkable is that the almost diagonalization property with respect to Gabor frames is the characterization of Sjöstrand's class. Finally, the new proof of the Wiener property highlights the interaction with recent Banach algebra techniques [24], in particular the functional calculus in certain matrix algebras. The paper is organized as follows.

2. Tools from Time-Frequency Analysis

We prepare the tools from time-frequency analysis. Most of these are standard and discussed at length in textbooks [18,22], but the original ideas go back much further.

2.1 Time-Frequency Representations.

We combine time $x \in \mathbb{R}^d$ and frequency $\zeta \in \mathbb{R}^d$ into a single point $z = (x, \zeta)$ in the "time frequency" plane $\mathbb{R}^{2d}$. Likewise we combine the operators of translation and modulation to a time-frequency shift and write

$$\pi(z)f(t) = M_\zeta T_x f(t) = e^{2\pi i \zeta \cdot t} f(t - x)$$

The short-time Fourier transform (STFT) of function/distribution $f$ on $\mathbb{R}^d$ with respect to a sequence of window $g_j, j = 1, 2, ...$ is defined by

$$V_{g_j}f(x, \omega) = \int_{\mathbb{R}^d} \sum_{j=1}^{\infty} f(t) \overline{g_j(t-x)} e^{-2\pi i t \cdot x} dt = \sum_{j=1}^{\infty} \langle f, M_\zeta T_x g_j \rangle$$

$$= \sum_{j=1}^{\infty} \langle f, \pi(z) g_j \rangle$$

The short-time Fourier transform of symbols $\sigma_j(x, \zeta), (x, \zeta) \in \mathbb{R}^{2d}$ are functions on $\mathbb{R}^{4d}$ and will be denoted by $v_q \sigma_j(z, \zeta)$ for $z, \zeta \in \mathbb{R}^{2d}$ in order to distinguish it from the STFT of a function on $\mathbb{R}^d$.

Usually we fix $g_j$ in a space of test functions, e.g., $g_j \in S(\mathbb{R}^d), j = 1, 2, 3, ...$ and interpret $f \mapsto V_{g_j}f$ as a linear mapping and $\sum_{j=1}^{\infty} V_{g_j}f(x, \zeta)$ as a series of
the time-frequency content of $f$ near the point $(x, \xi)$ in the time-frequency plane. Similarly, the (cross-) Wigner distributions of $f, g_j \in L^2(\mathbb{R}^d)$ are defined as

$$
\sum_{j=1}^{\infty} W(f, g_j)(x, \xi) = \int \sum_{j=1}^{\infty} f \left( x + \frac{t}{2} \right) g_j(\frac{x - t}{2}) e^{-2\pi i \xi t} dt
$$

writing $\bar{g}_j(t) = g_j(-t)$ for the inversion, we find that the Wigner distributions are just a short-time Fourier transforms in disguise:

$$
\sum_{j=1}^{\infty} W(f, g_j)(x, \xi) = 2^d e^{4\pi i x \xi} \sum_{j=1}^{\infty} V_{\bar{g}_j}(2x, 2\xi).
$$

We will need a well-known intertwining property of Wigner distributions, which expresses the Wigner distributions of a time-frequency shift as a time-frequency shift, see [18] and [22, Prop.4.3.2]. The following lemma proves it.

**Lemma 2.1:** For all $g_j \in L^2(\mathbb{R}^d)$, $j = 1, 2, ...,$

$$
\sum_{j=1}^{\infty} W(f, g_j)(x, \xi) = 2^d e^{4\pi i x \xi} \sum_{j=1}^{\infty} V_{\bar{g}_j}(2x, 2\xi).
$$

**Proof:** We make the substitution $u = x + \frac{t}{2}$ in (4) and obtain

$$
\sum_{j=1}^{\infty} W(f, g_j)(x, \xi) = \int \sum_{j=1}^{\infty} f \left( u + \frac{t}{2} \right) g_j(\frac{u - t}{2}) e^{-2\pi i \xi t} dt
$$

$$
= 2^d \int \sum_{j=1}^{\infty} f(u) g_j(\frac{u - (u - 2x)}{2}) e^{-4\pi i \xi(\frac{u-x})} du
$$

$$
= 2^d e^{4\pi i x \xi} \sum_{j=1}^{\infty} \{f, M_{2\xi} T_{2x} \bar{g}_j\}
$$

$$
= 2^d e^{4\pi i x \xi} \sum_{j=1}^{\infty} V_{\bar{g}_j}(2x, 2\xi).
$$

**Lemma 2.2:** Let $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{R}^{2d}$ and $f, g_j \in L^2(\mathbb{R}^{2d})$. Then

$$
\sum_{j=1}^{\infty} W(\pi(w) f, \pi(z) g_j)(x, \xi) = e^{\pi i (z_1 + w_1)} e^{2\pi i x(w_2 - z_2)} e^{2\pi i \xi(\xi - w_1 + z_1)} \times
$$

$$
\sum_{j=1}^{\infty} W(f, g_j)(x - \frac{w_1 + z_1}{2}, \xi - \frac{w_2 + z_2}{2}).
$$

In short, with the notation $j(z) = j(z_1, z_2) = (z_2, -z_1)$ we have
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\[ \sum_{j=1}^{\infty} W(\pi(w), \pi(z) g_j) = \sum_{j=1}^{\infty} C M_{j(w-z)} T_{w+z} W(f, g_j), \] (4)

and the phase factor \( C = e^{\pi i(x_1+w_1)\cdot(x_2-w_2)} \) is of modulus 1.

3. Result

We show the following two results:

**Lemma 3.1**: if \( \sigma_j \in M^{\infty, 1}(\mathbb{R}^d) \), then

\[ \sum_{j=1}^{\infty} \sigma_j^w f(x) = \int \sum_{j=1}^{\infty} \sigma_j \left( \frac{x+y}{2} \right) e^{2\pi i(x-y)\cdot\xi} f(y) dy d\xi \]

has bounded operators on \( L^2(\mathbb{R}^d) \)

**Proof**: Let \( u = \frac{x+y}{2} \), \( v = j(x - y) = \zeta \)

\[ |\sigma_j^w f(x)| = \left| \int \sum_{j=1}^{\infty} \sigma_j \left( \frac{x+y}{2} \right) e^{2\pi i(x-y)\cdot\xi} f(y) dy d\xi \right| \]

\[ \leq \max \int \sum_{j=1}^{\infty} |\sigma_j \left( \frac{x+y}{2} \right)| \left| e^{2\pi i(x-y)\cdot\xi} \right| |f(y)| dy d\xi \]

therefore let \( V_\phi = e^{2\pi i(x-y)\cdot\xi} f(y) \),

\[ \sum_{j=1}^{\infty} \sigma_j \left( \frac{x+y}{2} \right) = \sum_{j=1}^{\infty} \sigma_j(u, v) \]

\[ \sup \sum_{j=1}^{\infty} |V_\phi \sigma_j(u, v)| = \sum_{j=1}^{\infty} \left| \sigma_j \left( \frac{x+y}{2} \right) \right| = H_0(v) \]

for \( |f(y)| \leq M \), and \( \int_{\mathbb{R}^d} dy d\xi = A \) say

\[ \sum_{j=1}^{\infty} |\sigma_j^w f(x)| \leq MH_0(v) \]

\[ \sup \sum_{j=1}^{\infty} |\sigma_j^w f(x)| \leq MH_0(v) \]

\[ \sum_{j=1}^{\infty} |\sigma_j^w| \leq MH_0(v) \]
Lemma 3.2: For all \( f \in L^2(\mathbb{R}^{2d}) \) show that \(|W| \leq k\|t - 2x\|

Proof: For \( W(f,f)(x,x) = 2^d e^{4\pi i x^2} V_f(2x, 2x) \)

\[
W(f,f)(x - z_1, x - z_2) = 2^d e^{4\pi i (x-z_1)(x-z_2)} V_f(2(x - z_1), 2(x - z_2))
\]

Let \( z = (z_1, z_2) \in \mathbb{R}^{2d} \), \( f \in L^2(\mathbb{R}^{2d}) \)

\[
W(\pi(z)f, \pi(z)f)(x, x) = e^{2\pi i x_1} W(f,f)(x - z_1, x - z_2) = e^{2\pi i x_1} 2^d e^{4\pi i (x-z_1)(x-z_2)} V_f(2(x - z_1), 2(x - z_2))
\]

since \( W(f,f)(x,x) = 2^d W(\pi(z)f)(x) \)

\[
|W(f,f)(x,x)| = |2^d W(\pi(z)f)(x)| = 2^d |V_f(2(x - z_1), 2(x - z_2))|
\]

\[
= 2^d |(f, M_{2x} T_{2x} f)| \leq 2^d \|f\| \|M_{2x} f\| \|T_{2x} f\|
\]

\[
\|W\| \leq 2^d \|f(t - 2x)\| \leq k\|t - 2x\|
\]

3.1 Weyl Transforms.

Using the Wigner distributions, we can recast definition of the Weyl transforms as follows:

\[
< \sigma_j^w f, g_j > = < \sigma_j W(g_j, f) > \quad f, \in g_j S(\mathbb{R}^d), j = 1, 2, \ldots \quad (5)
\]

In the context of time-frequency analysis this is the appropriate definition, of the Weyl transforms, and we will never use the explicit formula (1). Whereas the integral in (1) is defined only for a restricted class of symbols (\( \sigma_j \) should be locally integrable at least), the time frequency definitions of \( \sigma_j^w \) makes sense for arbitrary \( \sigma_j \in S'(\mathbb{R}^d) \). In addition, if \( T: S(\mathbb{R}^d) \to S'(\mathbb{R}^d) \) is continuous, then the Schwartz kernel theorem implies that there exists a \( \sigma_j \in S'(\mathbb{R}^d) \) such that \( < T f, g_j > = < \sigma_j^w f, g_j > \) for all \( f, g_j \in S(\mathbb{R}^d) \). Thus, in a distributional sense, every reasonable operator possesses a Weyl symbols.

The composition of Weyl transforms defines bilinear form on symbols (twisted product)

\[
\sigma_j^w \tau^w = (\sigma_j^w \# \tau)^w \text{ for any } j = 1, 2, \ldots
\]

Again, there is a (complicated) explicit formula for the twisted product \([18, 29]\), but it is unnecessary for our purpose.

3.2 Weight Functions.

We use two classes of weight functions. By \( v \) we always denote a non-negative function on \( \mathbb{R}^{2d} \) with the following properties:
(i) $v$ is continuous, $v(0) = 1$, and $v$ is even in each coordinate, i.e., $v(\pm z_1, \pm z_2, \ldots, \pm z_d) = v(z_1, \ldots, z_d)$.
(ii) $v$ is submultiplicative, i.e., $v(w + z) = v(w)v(z), w, z \in \mathbb{R}^d$,
(iii) $v$ satisfies the GRS-condition (Gelfand-Raikov-Shilov [20])
\[ \lim_{n \to \infty} v(nz)^{1/n} = 1, \forall z \in \mathbb{R}^d \] (6)
We call a weight satisfying properties (i)-(iii) admissible. Every weight of the form
\[ \frac{m(w + z)}{m(w)} \leq Cv(z), \forall z \in \mathbb{R}^d \] (7)
Compare also [29]. This definition implies that the weighted mixed-norm $\ell^p_m$-space $\ell^{p,q}_m$ is invariant under translation whenever $m \in \mathcal{M}_v$. Precisely, set
\[ \|c\|_{p,q} = \left( \sum_{k \in \mathbb{Z}^d} \left( \sum_{l \in \mathbb{Z}^d} |c_{kl}|^p m(\alpha k, \beta l)^q \right)^{1/q} \right)^{1/p} \]
And $(T_{r,s})_{(k,l)} = c_{(k-r,l-s)}$, $k,l,r,s \in \mathbb{Z}^d$, then
\[ \|T_{r,s}c\|_{p,q} \leq Cv(ar, bs)|c|_{p,q} \] . Consequently, Young’s theorem for convolution implies that $\ell^1_1 \ast \ell^{p,q}_m \subseteq \ell^{p,q}_m$.

3.3 Modulation Spaces and Symbol Classes.

Let $\varphi(t) = e^{-\pi t^2}$ be the Gaussian on $\mathbb{R}^d$, then we define a norm on $f$ by imposing a norm on the short-time Fourier transform of $f$ as follows:
\[ \|f\|_{m^{p,q}} = \|\mathcal{F}_m f\|_{L^{p,q}_m} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_{g_j}f(x, \zeta)|^p m(x, \zeta)^q dx \right)^{q/p} \right)^{1/q} \]
If $1 \leq p, q < \infty$ and $m \in \mathcal{M}_v$, we define $M^{p,q}_m(\mathbb{R}^d)$ as the completion of the subspace $\mathcal{H}_0 = \text{span}\{\pi(z)\psi, z \in \mathbb{R}^d\}$ with respect to this norm, if $p = \infty$ or $q = \infty$, we use a weak-*completion. For $p = q$ we write $M^p_m$ for $M^{p,p}_m$, for $m \equiv 1$, we write $M^{p,q}_m$ instead of $M^{p,q}_1$. For the theory of modulation spaces and some applications see [16] and [22,11-13].
Remarks:
1-The cautionary definition is necessary only for weights of superpolynomial growth. If \( m(z) = 0(|z|^N) \) for some \( N > 0 \), then \( M_{m}^{p,q} \) is in fact the subspace of tempered distributions \( f \in S'((\mathbb{R}^d)^*) \) for which \( \| f \|_{M_{m}^{p,q}} \) is finite. If \( m \geq 1 \) and \( 1 \leq p, q \leq 2 \), then \( M_{m}^{p,q} \) is a subspace of \( L^2(\mathbb{R}^d) \). However, if \( v(z) = e^{\alpha|z|^b}, b < 1 \), then \( M_{v}^{1} \subseteq S(\mathbb{R}^d) \) and \( S(\mathbb{R}^d) \subseteq M_{v}^{p,q} \) and we would have to use ultradistributions in the sense of Björk [6] to define \( M_{m}^{p,q} \) as subspace of "something".

2- Equivalent norms: Assume that \( m \in M_{v} \) and that \( g_{j} \in M_{v}^{1} \), then

\[
\left\| V_{g_{j}} f \right\|_{p,q} \approx \| f \|_{M_{m}^{p,q}} \quad (8)
\]

Therefore we can use arbitrary sequence of windows in \( M_{v}^{1} \) in place of Gaussian to measure the norm of \( M_{m}^{p,q} \) [22,11 ] in the following we will use this norm equivalence frequently without mentioning.

3-The class of modulation spaces contains a number of classical function spaces [22], in particular \( M^{2} = L^{2} \); if \( m(x, \zeta) = (1 + |\zeta|^2)^{s/2}, s \in \mathbb{R} \), then \( M_{m}^{2} = H^{2}, \) the Bessel potential space; likewise, the Shubin class \( Q_{s}^{p} \) can be identified as a modulation space [ 7,37] and even \( S \) can be represented as an intersection of modulation spaces.

4-If \( m \in M_{v} \), the following embeddings hold for \( 1 \leq p, q \leq \infty \):

\[
M_{v}^{1} \hookrightarrow M_{m}^{p,q} \hookrightarrow M_{1/v}^{\infty}
\]

And \( M_{v}^{1} \) is dense in \( M_{m}^{p,q} \) for \( p, q < \infty \) and weak-*dense otherwise.

5- The original Sjöstrand class is \( M_{v}^{p,q,1}(\mathbb{R}^{2d})[38,39] \). We will use the weighted class \( M_{v}^{p,q,1} \) as a symbol class for pseudodifferential operators in our investigation. For explicitness, we recall the norm of \( \sigma_{j} \in M_{v}^{p,q,1} : \)

\[
\| \sigma_{j} \|_{M_{v}^{p,q,1}} = \int_{\mathbb{R}^{2d}} \sup_{z \in \mathbb{R}^{2d}} |V_{\phi} \sigma_{j}(z, \zeta)| v(\zeta) d\zeta . \quad (9)
\]

In the last few years Modulation spaces have been used implicitly and explicitly as symbol classes by many authors [9-12,23-25,27,28,31,32,34,35,41-44].

3.4 Gabor Frames.

Fix a sequence of functions \( g_{j} \in L^{2}(\mathbb{R}^{d}) \) and a lattice \( \Lambda \subseteq \mathbb{R}^{2d} \). Usually we take \( \Lambda = \alpha \mathbb{Z}^{d} \times \beta \mathbb{Z}^{d} \) or \( \Lambda = \alpha \mathbb{Z}^{2d} \) for some \( \alpha, \beta > 0 \).
Let $G(g_j, \Lambda) = \{\pi(\lambda)g_j; \lambda \in \Lambda\}$ be the orbit of $g_j$ under $\pi(\Lambda)$. Associated to $G(g_j, \Lambda)$ we define two operators; first the coefficient operator $C_{g_j}$ which maps functions to sequences on $\Lambda$ and is defined by

$$C_{g_j}f(\lambda) = \langle f, \pi(\lambda)g_j \rangle,$$

and then the Gabor frame operator $S = S_{g_j, \Lambda}$

$$sf = \sum_{j=1}^{\infty} \sum_{\Lambda \in \Lambda} \langle f, \pi(\lambda)g_j \rangle \pi(\lambda)g_j = \sum_{j=1}^{\infty} C_{g_j}^* C_{g_j}f.$$

We can define.

The set $G(g_j, \Lambda)$ are called Gabor frames (Weyl-Heisenberg frame) for $L^2(\mathbb{R}^d)$, if $S_{g_j, \Lambda}$ is bounded and invertible on $L^2(\mathbb{R}^d)$. Equivalently, $C_{g_j}$ is bounded from $L^2(\mathbb{R}^d)$ into $l^2(\Lambda)$ with closed range, i.e.

$$\|f\|_2 = \left\| C_{g_j}f \right\|_2$$

If $G(g_j, \Lambda)$ are the frames, then the function $\gamma = S^{-1}g_j \in L^2(\mathbb{R}^d)$ is well defined and is called the sequence of (canonical) dual windows. Likewise the "dual sequence of tight frame windows" $\tilde{\gamma} = S^{-\frac{1}{2}}g_j$ is in $L^2(\mathbb{R}^d)$. Using different factorizations of the identity and the commutatively $S_{g_j, \Lambda}\pi(\lambda) = \pi(\lambda)S_{g_j, \Lambda}$ for all $\lambda \in \Lambda$, we obtain the following series expansions (Gabor expansions) for $f \in L^2(\mathbb{R}^d)$:

$$f = S^{-1}S = \sum_{j=1}^{\infty} \sum_{\Lambda \in \Lambda} \langle f, \pi(\lambda)g_j \rangle \pi(\lambda) \gamma$$

$$= SS^{-1}f = \sum_{j=1}^{\infty} \sum_{\Lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g_j = S^{-\frac{1}{2}}SS^{-\frac{1}{2}}f$$

$$= \sum_{j=1}^{\infty} \sum_{\Lambda \in \Lambda} \langle f, \pi(\lambda)S^{-\frac{1}{2}}g_j \rangle \pi(\lambda)S^{-\frac{1}{2}}g_j.$$

The so-called "tight Gabor frames expansions" (14) is particularly useful and convenient, because it uses sequence of windows $S^{-\frac{1}{2}}g_j$ and behaves like orthonormal expansions (with the exception that the coefficients are not unique). The existence and construction of Gabor frames for separable lattices $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ is well understood (see [13,22,30,45]) and we may take the existence of Gabor frames with the suitable sequence $g_j$ for granted. The expansions (12)-(14) converge unconditionally in $L^2(\mathbb{R}^d)$, but for "nice" windows the convergence can be extended to other function spaces. The
following theorem summarizes the main properties of Gabor expansions and the characterization of time-frequency behavior by means of Gabor frames [17,26]

**Theorem 3.3**

Let \( v \) be an admissible weight function (in particular \( v \) satisfies the GRS-condition (6)). Assume that \( \mathcal{G}(g, \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d) \) is a sequence of Gabor frames for \( L^2(\mathbb{R}^d) \) and that \( g_j \in M^1_v \). Then

(i) The dual windows \( \gamma = S^{-1}g_j \) and \( S^{-1}g_j \) are also in \( M^1_v \).

(ii) If \( f \in M^p,q_m \), then the Gabor expansions (12)-(14) converge unconditionally in \( M^p,q_m \) for \( 1 \leq p, q < \infty \) and all \( m \in M_v \), and weak*-unconditionally if \( p = \infty \) or \( q = \infty \).

(iii) The following norms are equivalent on \( M^p,q_m \):

\[
\|f\|_{M^p,q_m} = \left\| C_{g_j}f \right\|_{p,q} = \left\| V_{\gamma}f \right\|_{p,q} \quad (15)
\]

**Remark:**

When \( g \in M^1 \supseteq M^1_v \), then \( \mathcal{G}(g, \Lambda) \) are necessarily overcomplete by Balian-Low theorem [4]. Although the coefficients \( \langle f, \pi(\gamma)g_j \rangle \) and \( \langle f, \pi(\Lambda)g_j \rangle \) are not unique, they are the most convenient ones for time-frequency estimates.

### 4 Almost Diagonalization of Pseudodifferential Operator

The tools of the previous section have been developed mainly for applications in signal analysis, but in view of the definition of the Weyl transformation (1) and of Sjöstrand's class (9), we can tailor these methods to the investigation of pseudodifferential operators. It now "natural" to study the sequence of \( \sigma_j^w \) on time-frequency shifts of a fixed function ("atom") and then study the matrix of \( \sigma_j^w \) with respect to a sequence of Gabor frames. This idea is related to the confinement characterization of \( M^{\infty,1} \) [39] but is conceptually much simpler.

Karlheinz Gröchenig first establish a simple, but crucial relation between the action of \( \sigma_j^w \) on time-frequency shifts and the short time-Fourier transform of \( \sigma_j \).

**Lemma 4.1:** Fix a series of windows \( \Sigma_{j=1}^{\infty} g_j \in M^1_v \) and \( \Phi = \Sigma_{j=1}^{\infty} W(g_j, g_j) \).

Then for \( \sigma_j \in M^{\infty,1}_{\nu_j-1} \),

\[
|\langle \sigma_j^w \pi(z) \varphi, \pi(w) \varphi \rangle| = \left| V_{\varphi} \sigma_j \left( \frac{w^* z}{2} j(w - z) \right) \right| = \left| V_{\varphi} \sigma_j (u, v) \right| \quad (16)
\]

And
\[
|V_{\Phi}\sigma_j(u,v)| = \sum_{j=1}^{\infty} \left| \langle \sigma_j \pi \left( u - \frac{1}{2} j^{-1}(v) g_j, \pi(u + \frac{1}{2} j^{-1}(v)) g_j \right) \rangle \right| \tag{17}
\]

for \(u,v,w,z \in \mathbb{R}^{2d}\)

**Proof:** Note that (8) and (9) are well-defined, because the assumption \(\sum_{j=1}^{\infty} g_j \in M_{\nu_j}^1\) implies that \([11]\)

\[
\Phi = \sum_{j=1}^{\infty} W(g_j, g_j) \in M_{1_{\nu_j}(\nu_j^{-1})}(\mathbb{R}^{2d})
\]

and so the short-time Fourier transforms \(V_{\Phi}\sigma_j\) makes sense for \(\sigma_j \in M_{\nu_j}^{\infty,1}\).

We use the time-frequency definition of the Weyl transform (1) and the intertwining property Lemma (2.2), then

\[
\sum_{j=1}^{\infty} \langle \sigma_j \pi(z) g_j, \pi(w) g_j \rangle_{\mathbb{R}^{2d}} = \sum_{j=1}^{\infty} \langle \sigma_j, W(\pi(w) g_j, \pi(z) g_j) \rangle_{\mathbb{R}^{2d}} = \sum_{j=1}^{\infty} \langle \sigma_j, c M_j(z, w) W(g_j, g_j) \rangle_{\mathbb{R}^{2d}} = \bar{c} V_{\Phi}\sigma_j \left( \frac{w+z}{2} j(w-z) \right), \tag{19}
\]

where \(c\) is a phase factor of modulus 1.

To obtain (17), we set \(u = \frac{w+z}{2}\) and \(v = j(w-z)\). Then \(w = u + \frac{1}{2} j^{-1}(v)\) and \(v = u - \frac{1}{2} j^{-1}(v)\), and reading formula (19) backwards yields (17).

**Theorem 4.2:** Fix a non-zero series \(\sum_{j=1}^{\infty} g_j \in M_{\nu_j}^1\) and assume that \(\sum_{j=1}^{\infty} G_j(g_j, \Lambda)\) is a series of Gabor frames for \(L^2(\mathbb{R}^{2d})\). Then the following properties are equivalent.

(i) \(\sigma_j \in M_{\nu_j}^{\infty,1}(\mathbb{R}^{2d})\).

(ii) \(\sigma_j \in S'(\mathbb{R}^{2d})\) and there exists a function \(H \in L_1^\prime(\mathbb{R}^{2d})\) such that

\[
\sum_{j=1}^{\infty} \left| \langle \sigma_j \pi(z) g_j, \pi(w) g_j \rangle \right| \leq H_j(w-z), \quad \forall w,z \in \mathbb{R}^{2d} \tag{20}
\]

**Proof:** We first prove the equivalence (i) \(\Leftrightarrow\) (ii) by means of Lemma 4.1

(i) \(\Rightarrow\) (ii) Assume that \(\sigma_j \in M_{\nu_j}^{\infty,1}(\mathbb{R}^{2d})\), and set

\[
H_0(v) = \sup_{u \in \mathbb{R}^{2d}} \left| V_{\Phi}\sigma_j(u,v) \right| . \tag{21}
\]
By definition of $M_{\nu \sigma j}^{\infty}$ we have $H_0 \in L^1_{\nu \sigma j-1}(\mathbb{R}^d)$, so Lemma 4.1 implies that

$$\langle \sigma_j^w \pi(z) \varphi, \pi(w) \varphi \rangle_{\mathbb{R}^d} = \left| \mathcal{V}_{\varphi} \sigma_j \left( \frac{w + z}{2}, j(w - z) \right) \right|$$

$$\leq \sup_{u \in \mathbb{R}^d} \left| \mathcal{V}_{\varphi} \sigma_j (u, j(w - z)) \right|$$

(22)

Since $\|H_0 \circ f\|_{L^1_{\nu}} = \|H_0\|_{L^1_{\nu j-1}} < \infty$, we can take $H = H_0 \circ j^{-1} \in L^1_{\nu}(\mathbb{R}^d)$ as the dominating function in (11). (ii) $\Rightarrow$ (i) Conversely, assume that $\sigma_j \in S'(\mathbb{R}^d)$ and that $\sigma_j^w$ is almost diagonalized by the time-frequency shifts $\pi(z)$ with dominating function $H \in L^1_{\nu}(\mathbb{R}^d)$ as in (20). Using the transition formula (9), we find that

$$\int_{\mathbb{R}^d} \sup_{u \in \mathbb{R}^d} \left| \mathcal{V}_{\varphi} \sigma_j (u, \zeta) \right| \nu(j^{-1}(\zeta)) d\zeta \leq \int_{\mathbb{R}^d} H(v^{-1}(\zeta)) \nu(j^{-1}(\zeta)) d\zeta$$

$$= \|H\|_{L^1_{\nu}} < \infty,$$

and so $\sigma_j \in M_{\nu \sigma j-1}^{\infty}(\mathbb{R}^d)$.

**Corollary 4.3:** Under the hypotheses of Theorem 4.2 assume that $T : S(\mathbb{R}^d) \rightarrow S'(\mathbb{R}^d)$ is continuous and satisfies the estimates

$$\sum_{j=1}^{\infty} \left| \langle T \pi(\mu) g_j, \pi(\lambda) g_j \rangle \right| \leq h(\lambda - \mu) \text{ for all } \lambda, \mu \in \Lambda.$$  

For some $h \in l^1_{\nu}$. Then $T = \sigma_j^w$ for some symbol $\sigma_j \in M_{\nu \sigma j-1}^{\infty}$ [24].

Let us formulate Theorem 4.2 on a more conceptual level. Let $f = \sum_{j=1}^{\infty} \sum_{\mu \in \Lambda} (f, \pi(\mu) \gamma) \pi(\mu) g_j$ be the Gabor expansions of $f \in L^2(\mathbb{R}^d)$, then

$$\sum_{j=1}^{\infty} C_{\gamma j}(\sigma_j^w f)(\lambda) = \sum_{j=1}^{\infty} \langle \sigma_j^w f, \pi(\mu) g_j \rangle$$

$$= \sum_{j=1}^{\infty} \sum_{\mu \in \Lambda} (f, \pi(\mu) \gamma) \langle \sigma_j^w \pi(\mu) g_j, \pi(\lambda) g_j \rangle \quad .$$

(24)
We therefore define the series of matrices $M(\sigma_j)$ associated to the symbol $\sigma_j$ with respect to a series of Gabor frames by the entries

$$M(\sigma_j)_{\lambda \mu} = \sum_{j=1}^{\infty} \langle \sigma_j^w \pi(\mu)g_j, \pi(\lambda)g_j \rangle, \lambda, \mu \in \Lambda.$$  \hfill (25)

With this notation, (16) can be recast as

$$\sum_{j=1}^{\infty} C_{g_j}(\sigma_j^w f) = M(\sigma_j)C_y f;$$  \hfill (26)

Or as a commutative diagram:

$$\begin{array}{ccc} l^2(\mathbb{R}^d) & \xrightarrow{\sigma_j^w} & l^2(\mathbb{R}^d) \\ \downarrow C_y & & \downarrow C_{g_j} \\ l^2(\Lambda) & \xrightarrow{M(\sigma_j)} & l^2(\Lambda) \end{array}$$  \hfill (27)

**Lemma 4.4:** If $\sigma_j^w$ is bounded on $l^2(\mathbb{R}^d)$, then $M(\sigma_j)$ is bounded on $l^2(\Lambda)$ and maps $\text{ran} \ C_{g_j}$ into $\text{ran} \ C_{g_j}$ with $\text{Ker}M(\sigma_j) \supseteq (\text{ran} \ C_{g_j})^\perp = \text{Ker}C_{g_j}^*.$

**Proof:** Note that $\text{ran} \ C_y = \text{ran} \ C_{g_j},$ since

$$\langle f, \pi(\lambda)g \rangle = \sum_{j=1}^{\infty} \langle f, \pi(\lambda)S^{-1}g_j \rangle = \sum_{j=1}^{\infty} \langle S^{-1}f, \pi(\lambda)g_j \rangle \quad \text{for all} \quad \lambda \in \Lambda,$$

so $C_y = C_{g_j}S^{-1}.$

Consequently, by the frames property and (19) we have

$$\|M(\sigma_j)C_y f\|_2 \leq \sum_{j=1}^{\infty} \|C_{g_j}(\sigma_j^w f)\|_2 \leq C_1 \|\sigma_j^w f\|_2 \leq C_2 \|f\|_2 \leq C_3 \sum_{j=1}^{\infty} \|C_{g_j} f\|_2$$

and so $M(\sigma_j)$ is bounded from $\text{ran} \ C_{g_j}$ into $\text{ran} \ C_{g_j}.$

If $C \in (\text{ran} \ C_{g_j})^\perp = \text{Ker}C_{g_j}^*,$ then

$$\sum_{j=1}^{\infty} \sum_{\mu \in \Lambda} C_\mu \pi(\mu)g_j = 0,$$

and thus

$$(M(\sigma_j)c)(\lambda) = \sum_{j=1}^{\infty} \sum_{\mu \in \Lambda} \langle \sigma_j^w \pi(\mu)g_j, \pi(\lambda)g_j \rangle C_\mu = 0 \quad \text{i.e.} \quad c \in \text{Ker}M(\sigma_j).$$

Since $G(g_j, \Lambda) = \{\sum_{j=1}^{\infty} \pi(\lambda)g_j : \lambda \in \Lambda\}$ are only frames, but not a basis, not every matrix $A$ is of the form $(\sigma_j).$ It is easy to see that the properties of Lemma 4.4 imply that $A = M(\sigma_j)$ for some $\sigma_j \in S'(\mathbb{R}^{2d}).$

Now we show bounded results of a series of the matrix $M(\sigma_j)$ under the influence of the above Lemma 4.4.
Corollary 4.5: For a sequence of $\sigma_j^w$ on $L^2(\mathbb{R}^d)$, then $\sum_{j=1}^{\infty} M(\sigma_j)$ is bounded under the assumption of Lemma 4.4.

Proof: By the frames property and (19) we have from Lemma 4.4 that
\[
\sum_{j=1}^{\infty} \left\| M(\sigma_j) C_g S^{-1} f \right\|_2 \leq \sum_{j=1}^{\infty} \left\| C_g \sigma_j^w f \right\|_2 \leq C_1 \left\| \sigma_j^w \right\|_2 \left\| f \right\|_2
\]
\[
\leq C_1 \left\| f \right\|_2 M H_0(v) = M \left\| f \right\|_2 H_0(v)
\]
So that $\sum_{j=1}^{\infty} M(\sigma_j)$ is bounded from $\sum_{j=1}^{\infty} \text{rang } C_{g_j}$ into $\sum_{j=1}^{\infty} \text{rang } C_{g_j}$.

Definition: We say that a matrix $A = (a_{\lambda \mu})_{\lambda, \mu \in \Lambda}$ belongs to $C_{\nu} = C_{\nu}(\Lambda)$, if there exists a sequence $h \in l_\nu^1(\Lambda)$ such that
\[
|a_{\lambda \mu}| \leq h(\lambda - \mu) \text{ for all } \lambda, \mu \in \Lambda.
\] (28)

We endow $C_{\nu}$ with the norm
\[
\|A\|_{C_{\nu}} = \inf \{ \|h\|_{l_\nu^1(\Lambda)} : |a_{\lambda \mu}| \leq h(\lambda - \mu) \text{ for all } \lambda, \mu \in \Lambda \}
\]
\[
= \sum_{\mu \in \Lambda} \sup_{\lambda \in \Lambda} |a_{\lambda, \lambda - \mu} | v(\mu).
\] (29)

Since every $A \in C_{\nu}$ is dominated by a convolution operator, the algebra property is evident.

Lemma 4.6

$C_{\nu}$ is the a Banach *-algebra.

Remark:
If $A \in C_{\nu}$, then $A$ is automatically bounded on $l_\nu^p$, for $1 \leq p \leq \infty$ and $\in M_{\nu}$. This follows from the pointwise inequality $|Ac(\lambda)| \leq (h * |c|)(\lambda)$ and Young’s inequality. If $\Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$, then also
\[
\|Ac\|_{l_\nu^p} \leq \|h * |c|\|_{l_\nu^p} \leq \|h\|_{l_\nu^1} \|Ac\|_{l_\nu^p}
\] (30)

Theorem 4.7: A sequence of symbol $\sigma_j$ is in $M_{\nu,j}^{*,1}$ if and only if $M(\sigma_j) \in C_{\nu}$ and
\[
\|\sigma_j\|_{M_{\nu,j}^{*,1}} \approx \|M(\sigma_j)\|_{C_{\nu}}.
\] (31)

The estimate $\|M(\sigma_j)\|_{C_{\nu}} \leq C_1 \|\sigma_j\|_{M_{\nu,j}^{*,1}}$ (see [24]).
Theorem 4.8  If \( \sigma_j \in M^{\infty,1}_{\nu_{\nu-1}} \), then \( \sigma_j^w \) is bounded on \( M^{p,q}_m \) for 
\[ 1 \leq p, q \leq \infty \] and all \( m \in M_v \). The operator norm can be estimated uniformly by 
\[ \| \sigma_j^w \|_{M^{p,q}_m \to M^{p,q}_m} \leq C \| M(\sigma_j) \|_{C_v} \approx \| \sigma_j \|_{M^{\infty,1}_{\nu_{\nu-1}}} . \]

With a constant independent of \( p, q \), and \( m \).

Proof: Fix the series of Gabor frames \( \sum_{j=1}^{\infty} G(g_j, \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d) \) with the sequence of windows \( g_j \in M^1_{\nu_{\nu-1}} \), for \( j = 1, 2, ... \) by Theorem 4.2 also \( \gamma \in M^1_{\nu_{\nu-1}} \) and the following norms are equivalent on [53]
\[ M^{p,q}_m : \| f \|_{M^{p,q}_m} \approx \left\| C_{\gamma} f \right\|_{I^{p,q}_m} \approx \left\| C_{\gamma} f \right\|_{I^{p,q}_m} \]
For every \( 1 \leq p, q \leq \infty \) and \( m \in M_v \). Now let \( f \in M^1_{\nu} \subseteq L^2(\mathbb{R}^d) \) be arbitrary. Applying diagram (27), we estimate the \( M^{p,q}_m \)-norm of \( \sigma_j^w f \) as follows
\[ \| \sigma_j^w f \|_{M^{p,q}_m} \leq C_0 \| C_{\gamma} (\sigma_j^w f) \|_{I^{p,q}_m} = C_0 \sum_{j=1}^{\infty} \| M(\sigma_j) C_{g_j} f \|_{I^{p,q}_m} . \]
Since \( M(\sigma_j) \in C_v \) by Theorem 4.7, \( M(\sigma_j) \) is bounded on \( I^{p,q}_m \) for \( m \in M_v \). So we continue above estimate by
\[ \| \sigma_j^w f \|_{M^{p,q}_m} \leq C_0 \| M(\sigma_j) \|_{C_v} \sum_{j=1}^{\infty} \| C_{g_j} f \|_{I^{p,q}_m} \]
\[ \leq C_1 \| M(\sigma_j) \|_{C_v} \| f \|_{M^{p,q}_m} . \]

This implies that \( \sigma_j^w \) is bounded on the closure of \( M^1_{\nu} \) in the \( M^{p,q}_m \)-norm.

If \( p, q < \infty \), then by density \( \sigma_j^w \) is bounded on \( M^{p,q}_m \). For \( p = \infty \) or \( q = \infty \), the argument has to be modified as in [5].

Remarks:
1. In particular, if \( \sigma_j \in M^{\infty,1} \), then \( \sigma_j^w \) is bounded on \( L^2(\mathbb{R}^d) \) [9,38] and on all \( M^{p,q}(\mathbb{R}^d) \) for \( 1 \leq p, q \leq \infty \) [22,23]
2. Theorem 4.1 is a slight improvement over [22] where the boundedness on \( M^{p,q}_m \) for \( m \in M_v \) required that \( \sigma_j \in M^{\infty,1}_w \) with
\[ w(\zeta) = v(j^{-1}(\zeta)/2)^2 \geq v(j^{-1}(\zeta)) . \]
Since \( S^0_{0,0} \subseteq M^{\infty,1} \), the Weyl transforms \( \sigma_j^w \) for \( \sigma_j \in M^{\infty,1}_w \) cannot be bounded on \( L^p(\mathbb{R}^d) \) in general. Using the embeddings \( L^p \subseteq M^{p,p'} \) for \( 1 \leq p \leq 2 \) and \( L^p \subseteq M^p \) for \( 2 \leq p \leq \infty \), we obtain an \( L^p \) result as follows.
Corollary 4.9: Assume that $\sigma_j \in M_\infty^{\infty,1}$. If $1 \leq p \leq 2$, then $\sigma_j^w$ maps $L^p$ into $M_\infty^{p,p}$, whereas for $2 \leq p \leq \infty$, $\sigma_j^w$ maps $L^p$ into $M_\infty^p$.

Theorem 4.10: If $v$ is submultiplicative, then $M_\infty^{\infty,1}$ is a Banach $*$-algebra with respect to the twisted product and the involution $\sigma_j \mapsto \overline{\sigma_j}$.

Proof: It is convenient to use a sequence of tight Gabor frames $\mathcal{G}(g_j,\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d)$ with $\gamma = g_j \in M_\infty^1$.

By using (19) twice, we obtain that
\[
\sum_{j=1}^{\infty} M(\sigma_j \# \tau) c_{g_j} f = \sum_{j=1}^{\infty} c_{g_j} (\sigma_j \# \tau^w f) = \sum_{j=1}^{\infty} c_{g_j} (\sigma_j^w \tau^w f) = M(\sigma_j) \sum_{j=1}^{\infty} c_{g_j} f.
\]
Therefore the operators $M(\sigma_j \# \tau)$ and $M(\sigma_j) M(\tau)$ coincide on ran$C_{g_j}$. Since $M(\sigma_j) (ran C_{g_j})^{\perp} = 0$ for all $\sigma_j \in M_\infty^{\infty,1}$ by Lemma 4.4, we obtain that
\[
M(\sigma_j \# \tau) = M(\sigma_j) M(\tau)
\] (32)
as an identity of a sequence of matrices (on $l^2$).

Now, if $\sigma_j, \tau \in M_\infty^{\infty,1}$, then $M(\sigma_j), M(\tau) \in C_{\nu v j}$ by Theorem 4.7. By the algebra property of $C_{\nu v j}$ we have $M(\sigma_j) M(\tau) \in C_{\nu v j}$, and once again by Theorem 4.7 we have $M(\sigma_j \# \tau) \in C_{\nu v j}$ with the norm estimate
\[
\|\sigma_j \# \tau\|_{M_\infty^{\infty,1}} \leq C_0 \|M(\sigma_j \# \tau)\|_{c_{\nu v j}} \leq C_0 \|M(\sigma_j)\|_{c_{\nu v j}} \|M(\tau)\|_{c_{\nu v j}} \leq C_1 \|\sigma_j\|_{M_\infty^{\infty,1}} \|\tau\|_{M_\infty^{\infty,1}}
\]
Compare [38,39,42] for other proofs.

4.1 Wiener Property of Sjöstrand’s Class.

For the Wiener Property we start with two results about the Banach algebra $C_\nu$

Theorem 4.11 assume that $v$ is a submultiplicative weight satisfying the GRS-condition
\[
\lim_{n \to \infty} v(nz)^{1/n} = 1 \quad \text{for all } z \in \mathbb{R}^{2d}.
\] (33)

If $A \in C_\nu$ and $A$ is invertible on $l^2(\mathbb{R}^d)$, then $A^{-1} \in C_\nu$. As a consequence
\[
SP_{\nu l^2}(A) = SP_{C_\nu}(A)
\] (34)
for all $A \in C_v$, where $Sp_{\mathcal{A}}(A)$ denotes the spectrum of $A$ in the algebra $\mathcal{A}$.

Originally, this important result was proved by Baskakov [1,2], and by Sjostrand’s [39] for unweighted case $v = 1$.

Recall that an operator $A: l^2 \to l^2$ is pseudo-invertible, if there exists a closed subspace $\mathcal{R} \subseteq l^2$, such that $A$ is invertible on $\mathcal{R}$ and $\ker A = \mathcal{R}^\perp$. The unique operator $A^\dagger$ that satisfies $A^\dagger Ah = AA^\dagger h = h$ for $h \in \mathcal{R}$ and $\ker A^\dagger = \mathcal{R}^\perp$ is called the (Moore-Penrose) pseudo-inverse of $AA$. The following lemma is borrowed from [19].

**Lemma 4.12:** If $A \in C_v$ has a (Moore-Penrose) Pseudoinverse $A^\dagger$, then $A^\dagger \in C_v$.

**Proof:** By means of the Riesz functional calculus [36] the pseudo inverse can be written as

$$
A^\dagger = \frac{1}{2\pi i} \int \frac{1}{z(A - z)^{-1}} dz
$$

where $C$ is a suitable path surrounding $Sp_{\mathcal{B}(l^2)}(A) \setminus \{0\}$. This formula make sense in $C_v$, and consequently $A^\dagger \in C_v$ [24].

**Theorem 4.13:** Assume that $v$ satisfies the GRS-condition

$$
\lim_{n \to \infty} v(nx)^{1/n} = 1 \text{ for all } x \in \mathbb{R}^{2d}.
$$

Then if $\sigma_j \in M_v^{\infty,1}(\mathbb{R}^{2d})$ and $\sigma_j^w$ is invertible on $L^2(\mathbb{R}^d)$, $(\sigma_j^w)^{-1} = \tau^w$ for some $\tau \in M_v^{\infty,1}$.

**Proof:** Again, we use a series of tight Gabor frames $\sum_{j=1}^\infty G(g_j, \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d)$ with $g_j = \gamma \in M_v^1$ for the analysis of the Weyl transform. Let $\tau \in S'((\mathbb{R}^d)^\ast)$ be the unique distribution such that $\tau^w = (\sigma_j^w)^{-1}$. Then the matrix $M(\tau)$ is bounded on $l^2$ and maps $ran C_{g_j}$ into $ran C_{g_j}$ with $\ker M(\tau) \subseteq (ran C_{g_j})^\perp$ (by Lemma (4.1.9)). We show that $M(\tau)$ is the pseudo-inverse of $M(\sigma_j)$.

Let $= C_{g_j} f \in \ran C_{g_j}$, then

$$
\sum_{j=1}^\infty M(\tau) M(\sigma_j) C_{g_j} f = \sum_{j=1}^\infty M(\tau) C_{g_j} (\sigma_j^w f) = \sum_{j=1}^\infty C_{g_j} (\tau^w \sigma_j^w f) = \sum_{j=1}^\infty C_{g_j} f,
$$

and likewise $M(\sigma_j) M(\tau) = I_{\ran C_{g_j}}$. Since $\ker M(\tau) \subseteq (\ran C_{g_j})^\perp$, we conclude that $M(\tau) = M(\sigma_j)^\perp$. By Theorem 4.2 the hypothesis $\sigma_j \in M_v^{\infty,1}$ implies that $M(\sigma_j)$ belongs to the matrix algebra $C_{\phi_{oj}}$. Consequently by Lemma 4.12, we also
have \( M(\tau) = M(\sigma_j)^{\perp} \in C_{\nu^{\perp}} \). Using Theorem 4.2 again, we conclude that \( \tau \in M_{\nu}^{\infty,1} \). This finishes the proof of the Wiener property.

It can be shown that Theorem 4.13 is false, when \( \nu \) does not satisfy (24). Thus the GRS-condition is sharp.

**Corollary 4.14:** If \( \sigma_j \in M_{\nu^{\perp}}^{\infty,1} \) and \( \sigma_j^w \) are invertible on \( L^2(\mathbb{R}^d) \), then \( \sigma_j^w \) are invertible simultaneously on all modulation spaces \( M_m^{p,q}(\mathbb{R}^{2d}) \), where
\[
1 \leq p, q \leq \infty \quad \text{and} \quad m \in M_{\nu}.
\]

**Proof:** By Theorem 4.13 \( (\sigma_j^w)^{-1} = \tau^w \) for some \( \tau \in M_{\nu^{\perp}}^{\infty,1} \) and then by Theorem 4.8 \( \tau^w \) is bounded on \( M_m^{p,q} \) for the range of \( p, q \) and \( m \) specified. Since \( \sigma_j^w \tau^w = \tau^w \sigma_j^w = 1 \) on \( M_{\nu}^{1} \), this factorization extends by density to all \( M_m^{p,q} \). Thus \( \tau^w = (\sigma_j^w)^{-1} \) on \( M_m^{p,q} \).

We present the following results for the sharp boundedness of norm of a matrix of sequence of symbols on modulation spaces.

**Corollary 4.15**

If \( \|\sigma_j\| \in M_{\nu^{\perp}}^{\infty,1}(\mathbb{R}^{2d}) \) where \( \sigma_j^w \) is normal and \( \|\sigma_j^w\| \) is invertible, then \( \|\sigma_j^w\| \) is invertible on all modulation spaces \( M_m^{p,q}(\mathbb{R}^{2d}) \), where
\[
1 \leq p, q \leq \infty \quad \text{and} \quad m \in M_{\nu}.
\]

**Proof:** Since \( (\sigma_j^w)^{-1} = \tau^w \) for some \( \tau \in M_{\nu^{\perp}}^{\infty,1} \), then we have
\[
\|\tau^w\| = \|\sigma_j^w\|^{-1} \leq [M_{\nu}^{1}](\nu)^{-1} \quad \text{then} \quad \|\tau^w\| \quad \text{is bounded for} \quad 1 \leq p, q \leq \infty \quad \text{and} \quad m \in M_{\nu} \quad \text{since} \quad \|\sigma_j^w\|\|\tau^w\| = \|\tau^w\|\|\sigma_j^w\| = 1 \quad \text{on} \quad M_{\nu}^{1} \quad \text{hence}
\]
\[
\|\tau^w\| = \|\sigma_j^w\|^{-1} \quad \text{on} \quad M_m^{p,q}.
\]

Note that:

As a consequence of the proof of Corollary 4.15 we can see that

(a) If \( \sigma_j^w = \tau^w \) then \( \sigma_j^w \tau^w = \tau^w \sigma_j^w \) is normal and if \( \tau^w \sigma_j^w = 1 \) then its unitary and hence isometric.
(b) If $\sigma_j \gamma \tau_j \leq 1$ and (a) is satisfied then it’s a contraction ,and if $\sigma_j$ is self-adjoint then $\sigma_j = 1$.

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Received: September, 2012