$L^{p_1} - L^{p_2}$ Version of Miyachi’s Theorem for the $q$-Dunkl Transform on the Real Line

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Abstract

This paper deals with Miyachi’s theorem for the $q$-Dunkl Transform introduced and studied in [3].

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1 Introduction

In harmonic analysis, the uncertainty principle states that a function and its Fourier transform can not simultaneously decrease very quickyl. In the literature, this fact is in general given by the way of some inequalities involving a function $f$ and its Fourier transform $\hat{f}$. One of the famous formulations of the uncertainty principle is stated by the so-called Hardy’s theorem (see [8, 12, 11]), which assert that if

$$\sup_{x \in \mathbb{R}} |e^{ax^2} f(x)| < \infty, \quad \sup_{\lambda \in \mathbb{R}} |e^{b\lambda^2} \hat{f}(\lambda)| < \infty \quad \text{and} \quad ab > \frac{1}{4},$$

then $f \equiv 0$. 
In [4], M. G. Cowling and J. F. Price obtained and $L^p$ version of Hardy’s theorem by showing that for $p, n \in [1, +\infty]$ with at least one of them is finite, if \[ \| e^{ax^2} f(x) \|_p < +\infty, \| e^{a\lambda^2} \hat{f}(\lambda) \|_n < +\infty \text{ and } ab > \frac{1}{4}, \] then $f = 0$.

Generalization of these results in both classical and quantum analysis have been revealed (see [2, 6, 9, 5, 17, 18]) and many versions of Hardy uncertainty principles were obtained for several generalized Fourier transforms.

In [3], Bettaibi et al. introduced and studied a $q$-analogue of the classical Bessel-Dunkl transform ($q$-Dunkl transform). In particular they provided, for this transform, a Plancherel formula and proved an inversion theorem. In this paper, we state an $L^{p_1} + L^{p_2}$ version of Miyachi’s Theorem for the $q$-Dunkl transform $F_{D}^{\alpha,q}(f)$.

This paper is organized as follows: in Section 2, we present some preliminaries notions and notations useful in the sequel. In Section 3, we recall and state some results and properties from the theory of the $q$-Dunkl operator and the $q$-Dunkl transform (see [3]). Section 4 is devoted to give an $L^{p_1} + L^{p_2}$ version of Miyachi’s Theorem for the $q$-Dunkl transform $F_{D}^{\alpha,q}$. Section 5 by the application of the of Miyachi’s Theorem proved in section 4, we deduce the Miyachi’s theorem for the $q^2$-analogue Fourier transform and a corollary for the $q$-Bessel transform.

2 Notations and preliminaries

Throughtout this paper, we assume $q \in ]0, 1[$, we refer to the general reference [10] for the definitions, notations and properties of the $q$-shifted factorials and the $q$-hypergeometric functions.

We write $\mathbb{R}_q = \{ \pm q^n : n \in \mathbb{Z} \}, \mathbb{R}_{q,+} = \{ q^n : n \in \mathbb{Z} \}$,

\[ [x]_q = \frac{1 - q^x}{1 - q}, \quad x \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q,q)_n}{(1-q)^n}, \quad n \in \mathbb{N}. \tag{1} \]

The $q^2$-analogue differential operator is (see [15, 16])

\[ \partial_{q} f(z) = \begin{cases} \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) + 2f(-z)}{2(1-q)^2} & \text{if } z \neq 0 \\ \lim_{x \to 0} \partial_{q} f(x) (\text{in } \mathbb{R}_q) & \text{if } z = 0 \end{cases} \tag{2} \]

Remark that if $f$ is differentiable at $z$, then $\lim_{q \to 1} \partial_{q} f(x) = f'(z)$. A repeated application of the $q^2$-analogue differential operator is denoted by:

\[ \partial_{q}^0 f = f, \quad \partial_{q}^{n+1} f = \partial_{q} (\partial_{q}^n f). \]
The following lemma lists some useful computational properties of $\partial_q$.

**Lemma 2.1**  
1. For all function $f$ on $\mathbb{R}_q$, \( \partial_q f(z) = f(ze^{-1} + f(ze^{1})) \).  
2. For two functions $f$ and $g$ on $\mathbb{R}_q$, we have  
   - if $f$ is even and $g$ is odd,  
     \[ \partial_q(fg)(z) = q\partial_q(f)(qz)g(z) + f(qz)\partial_q(g)(z) = \partial_q(g)(z)f(z) + qg(qz)\partial_q(f)(qz); \]  
   - if $f$ and $g$ are even,  
     \[ \partial_q(fg)(z) = \partial_q(f)(q^{-1}z) + f(z)\partial_q(g)(z). \]

Here, for a function $f$ defined on $\mathbb{R}_q$, $f_e$ and $f_o$ are its even and odd parts respectively.

**Proof:** The proof is straightforward (we refer to [3]) □

The operator $\partial_q$ induces a $q$-analogue of the classical exponential function (see [15, 16])

\[ e(z; q^2) = \sum_{n=0}^{+\infty} \frac{z^n}{[n]_q!}, \quad \text{with} \quad a_{2n} = a_{2n+1} = q^{n(n+1)}. \]  

The $q$-jackson integrals are defined by (see [13])

\[ \int_0^a f(x)d_qx = (1-q)a \sum_{n=0}^{+\infty} q^n f(aq^n), \quad \int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx, \]

\[ \int_0^{+\infty} f(x)d_qx = (1-q) \sum_{n=-\infty}^{+\infty} q^n f(q^n), \]

and

\[ \int_{-\infty}^{+\infty} f(x)d_qx = (1-q) \sum_{n=-\infty}^{+\infty} q^n f(q^n) + (1-q) \sum_{n=-\infty}^{+\infty} q^n f(-q^n) \]

provided the sums converge absolutely.

The $q$-gamma function is given by (see [13])

\[ \Gamma_q(x) = \frac{\Gamma(q; q)_\infty}{(q^x, q)_\infty} (1-q)^{1-x}, \quad x \neq 0, -1, -2, ... \]

In what follows, we will need the following sets and spaces:
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- \( L^\infty_q(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup_{x \in \mathbb{R}_q} |f(x)| < +\infty \right\} \).

- \( L^p_{\alpha,q}(\mathbb{R}_q) = \left\{ f : \|f\|_{p,\alpha,q} = \left( \int_{-\infty}^{+\infty} |f(x)|^p |x|^{2\alpha+1} d_q x \right)^{\frac{1}{p}} < +\infty \right\} \).

3 The \( q \)-Dunkl operator and the \( q \)-Dunkl transform

In this section we collect some basic properties of the \( q \)-Dunkl operator and the \( q \)-Dunkl transform introduced in [3], useful in the sequel.

For \( \alpha \geq -\frac{1}{2} \), the \( q \)-Dunkl operator is defined by

\[
\Lambda_{\alpha,q}(f)(x) = \partial_q[H_{\alpha,q}(f)](x) + [2\alpha + 1]_q \frac{f(x) - f(-x)}{2x},
\]

where

\[
H_{\alpha,q} : f = f_e + f_o \mapsto f_e + q^{2\alpha+1}f_o.
\]

It was shown in [3] that for each \( \lambda \in \mathbb{C} \), the function

\[
\psi_{\lambda}^{\alpha,q}(x) := j_\alpha(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha + 2]_q} j_{\alpha+1}(\lambda x; q^2)
\]

is the unique solution of the \( q \)-diffentiel equation:

\[
\left\{ \begin{array}{l}
\Lambda_{\alpha,q}(f) = i\lambda f \\
 f(0) = 1,
\end{array} \right.
\]

where \( j_\alpha(\cdot; q^2) \) is the normalized third Jackson’s \( q \)-Bessel function given by

\[
j_\alpha(x; q^2) = \sum_{n=0}^{+\infty} (-1)^n \frac{q^{n(n+1)}}{(q^2; q^2)_n (q^{2(\alpha+1)}; q^2)_n} ((1 - q)x)^{2n}.
\]

The function \( \psi_{\lambda}^{\alpha,q} \), has an unique extension to \( \mathbb{C} \times \mathbb{C} \) and we have the following properties.

- \( \psi_{a\lambda}^{\alpha,q}(x) = \psi_{\lambda}^{\alpha,q}(ax) = \psi_{\overline{a}\lambda}^{\alpha,q}(\lambda), \forall a, x, \lambda \in \mathbb{C}. \)
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For \( \alpha = -\frac{1}{2} \), \( \psi_{a\lambda}^{\alpha,q}(x) = e(i\lambda x; q^2) \) and for \( \alpha > -\frac{1}{2} \), \( \psi_{a\lambda}^{\alpha,q} \) has the following \( q \)-integral representation of Mehler type

\[
\psi_{a\lambda}^{\alpha,q}(x) = \frac{(1 + q)\Gamma_{q^2}(\alpha + 1)}{2\Gamma_{q^2}(\alpha + \frac{1}{2})} \int_{-1}^{1} \frac{(t^2q^2; q^2)_{\infty}}{(t^2q^{2\alpha+1}; q^2)_{\infty}} (1 + t)e(i\lambda xt; q^2) dt.
\]

(6)

For all \( x, \lambda \in \mathbb{R}_q \),

\[
|\psi_{\lambda}^{\alpha,q}(x)| < \frac{4}{(q; q)_\infty}.
\]

(7)

The \( q \)-Dunkl transform \( F_{D}^{\alpha,q} \) is defined on \( L_{1,a,q}^1(\mathbb{R}_q) \) (see [3]) by

\[
F_{D}^{\alpha,q}(f)(\lambda) = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} f(x)\psi_{-\lambda}^{\alpha,q}(x)|x|^{2\alpha+1}d_qx,
\]

where

\[
c_{\alpha,q} = \frac{(1 + q)^{-\alpha}}{\Gamma_{q^2}(\alpha + 1)}.
\]

It satisfies the following properties:

• For \( \alpha = -\frac{1}{2} \), \( F_{D}^{\alpha,q} \) is the \( q^2 \)-analogue Fourier transform \( \hat{f}(.; q^2) \) given (see [15, 16])

\[
\hat{f}(\lambda; q^2) = \frac{(1 + q)^{\frac{1}{2}}}{2\Gamma_{q^2}(\frac{1}{2})} \int_{-\infty}^{+\infty} f(x)e(-i\lambda x; q^2) d_qx.
\]

• On the even functions space, \( F_{D}^{\alpha,q} \) coincides with the \( q \)-Bessel transform given by (see [3])

\[
\mathcal{F}_{\alpha,q}(f)(\lambda) = c_{\alpha,q} \int_{0}^{+\infty} f(x)j_{\alpha}(\lambda x; q^2)x^{2\alpha+1}d_qx.
\]

• For all \( f \in L_{1,a,q}^1(\mathbb{R}_q) \), we have:

\[
\|F_{D}^{\alpha,q}\|_{\infty,q} \leq \frac{2c_{\alpha,q}}{(q; q)_\infty} \|f\|_{1,a,q}.
\]

(8)

the \( q \)-Dunkl transform \( F_{D}^{\alpha,q} \) is an isomorphism from \( L_{a,q}^2(\mathbb{R}_q) \) onto itself and satisfies the following Plancherel formula and the inversion theorem:

\[
\forall f \in L_{a,q}^2(\mathbb{R}_q), \|F_{D}^{\alpha,q}(f)\|_{2,a,q} = \|f\|_{2,a,q}
\]

(9)

and

\[
(F_{D}^{\alpha,q})^{-1}(f)(\lambda) = F_{D}^{\alpha,q}(f)(-\lambda), \; \lambda \in \mathbb{C}.
\]
4 $L^p_1 - L^p_2$-version of Miyachi's Theorem for the $q$-Dunkl Transform on the Real Line

In this section, we shall state an $L^p_1 - L^p_2$-version of Miyachi's Theorem for the $q$-Dunkl Transform $F_{D}^{\alpha,q}$. We begin by the two following lemmas. The first is a deduction from the Mehler type relation (6) and ([2], Proposition 1) and the second proved by A. Miyachi in [14] (Proposition 4).

**Lemma 4.1** for all $\lambda \in \mathbb{C}$ and all $x \in \mathbb{R}$, we have

$$|\psi_{\lambda}^{\alpha,q}(x)| < 2e^{k|x\lambda|},$$

with $k = 1 + \sqrt{q}$.

**Lemma 4.2** Suppose $f$ is an entire function on $\mathbb{C}$ and suppose there exist constants $A, B > 0$ such that:

$$|f(z)| \leq Ae^{B(z)} \text{ for all } z \in \mathbb{C}.$$

Also suppose

$$\int_{-\infty}^{+\infty} \log^+ |f(t)|dt < +\infty.$$

Where $\log^+ x = \log x$ if $x > 1$ and $\log^+ x = 0$ if $x \leq 1$, then $f$ is a constant function.

**Lemma 4.3** Let $h$ be an entire function on $\mathbb{C}$ and $\alpha \geq -\frac{1}{2}$, such that:

$$\forall z \in \mathbb{C}, |h(z)| \leq Ae^{B(z)}, \text{ for some constants } A, B > 0.$$

and

$$\int_{-\infty}^{+\infty} \log^+ |h(t)||t|^{2\alpha+1}dt < +\infty.$$

Then $h$ is a constant function.

**Proof:** we have

$$\int_{-\infty}^{+\infty} \log^+ |h(t)|dt = \int_{-1}^{+1} \log^+ |h(t)|dt + \int_{|t|>1} \log^+ |h(t)|dt \quad (10)$$
and

\[
\int_{|t|>1} \log^+ |h(t)|dt \leq \int_{|t|>1} \log^+ |h(t)||t|^{2\alpha+1}dt < +\infty \quad (11)
\]

\( h \) is entire, particularly is continuous, then \( \int_{-1}^{+1} \log^+ |h(t)|dt < +\infty \), we deduce from (10) and (11) that \( \int_{-\infty}^{+\infty} \log^+ |h(t)|dt < +\infty \) the Lemma 4.2 finishes the proof.

**Theorem 4.4** Let \( a > 0, p_1, p_2 \in [1, +\infty] \) and \( f \) be a function defined on \( \mathbb{R}_q \) such that

\[
e^{|ax^2|}f \in L_{\alpha,q}^{p_1}(\mathbb{R}_q) + L_{\alpha,q}^{p_2}(\mathbb{R}_q)
\]

(12)

Then \( F_D^{\alpha,q}(f) \) is entire on \( \mathbb{C} \) and for all \( b \in ]0, a[ \), we have:

\[
\forall z \in \mathbb{C}, |F_D^{\alpha,q}(f)(z)| \leq Ce^{\frac{k^2|z|^2}{|b|}}
\]

for some positive constant \( C \).

**Proof:** Since \( |\psi_{\alpha,q}(x)| \leq 2e^{1+\sqrt{q|x^2}|} \), for all \( x \in \mathbb{C} \) and \( t \in \mathbb{R} \), then from the hypothesis, the Hölder’s inequality and the analyticity theorem one deduce that \( F_D^{\alpha,q}(f) \) is entire on \( \mathbb{C} \), (12) implies that there exists \( u_1, u_2 \in L_{\alpha,q}^{p_j}(\mathbb{R}_q) \) such that \( e^{|ax^2|}f(x) = u_1(x) + u_2(x) \) and for all \( z \in \mathbb{C} \), we have for \( j = 1, 2 \)

\[
|F_D^{\alpha,q}(f)(z)| \leq \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} |\psi_{\alpha,q}(x)||f(x)||x|^{2\alpha+1}d_qx
\]

\[
\leq c_{\alpha,q} \int_{-\infty}^{+\infty} e^{k|z||x|-ax^2}e^{ax^2}|f(x)||x|^{2\alpha+1}d_qx
\]

\[
\leq c_{\alpha,q} \sum_{j=1}^{l} \int_{-\infty}^{+\infty} e^{k|z||x|-ax^2}|u_j(x)||x|^{2\alpha+1}d_qx
\]

\[
\leq c_{\alpha,q} \sum_{j=1}^{l} \left( \int_{-\infty}^{+\infty} e^{n_j(k|z||x|-ax^2)|x|^{2\alpha+1}d_qx} \right)^{\frac{1}{n_j}} \|u_j\|_{p_j,\alpha,q},
\]

where \( n_1, n_2 \) is the real satisfying \( \frac{1}{p_1} + \frac{1}{n_1} = \frac{1}{p_2} + \frac{1}{n_1} = 1 \).

Now, for \( b \in ]0, a[ \), we have

\[
\left( \int_{-\infty}^{+\infty} e^{n_j(k|z||x|-ax^2)|x|^{2\alpha+1}d_qx} \right)^{\frac{1}{n_j}} = \left( \int_{-\infty}^{+\infty} e^{n_j(k|z||x|-ax^2)}e^{-n_j(a-b)x^2}|x|^{2\alpha+1}d_qx \right)^{\frac{1}{n_j}}
\]

\[
\left( \int_{-\infty}^{+\infty} e^{n_j(k|z||x|-ax^2)} \right)^{\frac{1}{n_j}} = \left( \int_{-\infty}^{+\infty} e^{-n_j(a-b)x^2} \right)^{\frac{1}{n_j}}
\]

\[
\left( \int_{-\infty}^{+\infty} e^{-n_j(a-b)x^2} \right)^{\frac{1}{n_j}} = \left( \int_{-\infty}^{+\infty} e^{-n_j(a-b)x^2} \right)^{\frac{1}{n_j}}
\]

\[
\left( \int_{-\infty}^{+\infty} e^{-n_j(a-b)x^2} \right)^{\frac{1}{n_j}} = \left( \int_{-\infty}^{+\infty} e^{-n_j(a-b)x^2} \right)^{\frac{1}{n_j}}
\]

\[
\left( \int_{-\infty}^{+\infty} e^{-n_j(a-b)x^2} \right)^{\frac{1}{n_j}} = \left( \int_{-\infty}^{+\infty} e^{-n_j(a-b)x^2} \right)^{\frac{1}{n_j}}
\]

\[
\left( \int_{-\infty}^{+\infty} e^{-n_j(a-b)x^2} \right)^{\frac{1}{n_j}} = \left( \int_{-\infty}^{+\infty} e^{-n_j(a-b)x^2} \right)^{\frac{1}{n_j}}
\]
\[ \leq \left( \sup_{x \in [0, +\infty]} e^{n_j(k|x-ax^2|)} \right)^{\frac{1}{nj}} \left( \int_{-\infty}^{+\infty} e^{-n_j(a-b)x^2} |x|^{2\alpha+1} d_q x \right)^{\frac{1}{nj}} \]

\[ = C_j e^{k^2|x|^2}, \]

with \( C_j = \left( \int_{-\infty}^{+\infty} e^{-n_j(a-b)x^2} |x|^{2\alpha+1} d_q x \right)^{\frac{1}{nj}}. \)

Then we have \( |F^\alpha_q(D(f)(z))| \leq C e^{k^2|x|^2} \) with \( C = c_{\alpha,q} (C_1 + C_2). \)

**Theorem 4.5** Let \( a, b > 0 \) such that \( ab < \frac{k^2}{4} \) where \( k = \sqrt{q} + 1 \) and \( \lambda \in \mathbb{C} \), for \( a < t < \frac{k^2}{4b} \) we put

\[ h_t^{\alpha,q}(x) = \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} e^{-\frac{k^2}{4t} \lambda^2} \psi^{\alpha,q}_\lambda(x) |\lambda|^{2\alpha+1} d_q \lambda \]

then we have:

- There exists \( D > 0 \) and \( E_t > 0 \) depend on \( t \) such that:
  \[ |e^{ax^2} h_t^{\alpha,q}(x)| \leq E_t e^{-Dx^2} \]

- For \( p \in [1, +\infty] \)
  \[ e^{ax^2} h_t^{\alpha,q}(x) \in L^p_{\alpha,q}(\mathbb{R}_q) \]

**Proof:** We have by the use of Lemma (4.1) and we not lose generality if we assume that \( x \geq 0, \) for a very small \( \varepsilon > 0 : \)

\[ |h_t^{\alpha,q}(x)| \leq \frac{c_{\alpha,q}}{2} \int_{-\infty}^{+\infty} e^{-\frac{k^2}{4t} \lambda^2} |\psi^{\alpha,q}_\lambda(x)||\lambda|^{2\alpha+1} d_q \lambda \]

\[ \leq c_{\alpha,q} \int_{-\infty}^{+\infty} e^{-\frac{k^2}{4t} \lambda^2} e^{k|\lambda x|} |\lambda|^{2\alpha+1} d_q \lambda \]

\[ \leq 2c_{\alpha,q} \int_{0}^{+\infty} e^{(k\lambda x - \frac{k^2}{4t} \lambda^2)} \lambda^{2\alpha+1} d_q \lambda \]

\[ \leq 2c_{\alpha,q} \sup_{\lambda \in [0, +\infty]} \left( e^{(k\lambda x + (\varepsilon - \frac{k^2}{4t}) \lambda^2)} \right) \int_{0}^{+\infty} e^{-\varepsilon \lambda^2} \lambda^{2\alpha+1} d_q \lambda \]

\[ \leq 2c_{\alpha,q} e^{\left( \frac{3\varepsilon^2}{4(\varepsilon - \frac{k^2}{4t})^2} \right)} \int_{0}^{+\infty} e^{-\varepsilon \lambda^2} \lambda^{2\alpha+1} d_q \lambda \]
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we put $M = \int_0^{+\infty} e^{-\varepsilon \lambda^2} \lambda^{2\alpha+1} d\lambda$ then we have

$$|e^{a x^2} h_t^{\alpha,q}(x)| \leq 2Mc_{\alpha,q} e^{\left(a + \frac{3k^2}{4(\varepsilon - \frac{k^2}{4})}\right)x^2}$$

for a very small $\varepsilon > 0$ we have $a + \frac{3k^2}{4(\varepsilon - \frac{k^2}{4})} < 0$ because $a < t$, which finish the proof of the first statement.

-We have $|e^{a x^2} h_t^{\alpha,q}(x)| \leq E_t e^{-Dx^2}$ for $D, E_t > 0$ then we conclude that

$$e^{a x^2} h_t^{\alpha,q}(x) \in L_{p,q}^p(\mathbb{R})$$

for $p \in [1, +\infty]$ ■

**Theorem 4.6** Let $a, b > 0$, $p_1, p_2 \in [1, +\infty]$ where $i = 1, ..., l$ and $f$ be a function defined on $\mathbb{R}_q$ such that:

$$e^{a x^2} f \in L_{p,q}^{p_1}(\mathbb{R}) + L_{p,q}^{p_2}(\mathbb{R})$$

and

$$\int_{-\infty}^{+\infty} \log^+ \left[ \frac{\mathcal{F}^{\alpha,q}_D(f)(x)e^{bx^2}}{\lambda} \right] |x|^{2\alpha+1} dx < +\infty$$

for some $\lambda > 0$, then

- if $ab > \frac{k^2}{4}$ then $f = 0$ on $\mathbb{R}_q$.
- if $ab = \frac{k^2}{4}$ then $f = N h^{\alpha,q}_a$ where $N$ a constant and $|N| \leq \lambda$.
- if $ab < \frac{k^2}{4}$ then there exists many functions.

**Proof:** Let $a, b > 0$ and $h$ be the function on $\mathbb{C}$ defined by:

$$h(z) = e^{\frac{k^2}{4z^2}} \mathcal{F}^{\alpha,q}_D(f)(z)$$

- **Case 1:** ($ab > \frac{k^2}{4}$)

from Theorem 4.4 we deduce that the function $h$ is entire, on other hand, we note that for a very small $\varepsilon > 0$ we have $a - \varepsilon \in [0, a]$:
∀z ∈ C, |h(z)| ≤ Ce^{\frac{k^2}{4(\alpha - \epsilon)}}(\text{Re}(z))^2 \tag{17}

and

\begin{align*}
\int_{-\infty}^{+\infty} \log^+ |h(x)||x|^{2\alpha+1}dx &= \int_{-\infty}^{+\infty} \log^+ |e^{\frac{k^2}{4\pi}x^2} F_D^{\alpha,q}(f)(x)||x|^{2\alpha+1}dx \\
&= \int_{-\infty}^{+\infty} \log^+ \left( \lambda e^{\frac{k^2}{4\pi}x^2 - b} e^{bx^2} F_D^{\alpha,q}(f)(x) \right) |x|^{2\alpha+1}dx
\end{align*}

≤ \int_{-\infty}^{+\infty} \log^+ \lambda e^{\frac{k^2}{4\pi}x^2 - b} x^{2\alpha+1}dx + \int_{-\infty}^{+\infty} \log^+ e^{bx^2} \left| \frac{F_D^{\alpha,q}(f)(x)}{\lambda} \right| |x|^{2\alpha+1}dx

because \log^+(\varpi \rho) ≤ \log^+(\varpi) + \rho for all \varpi, \rho > 0, (15) implies that:

\begin{equation}
\int_{-\infty}^{+\infty} \log^+ |h(x)||x|^{2\alpha+1}dx < +\infty. \tag{18}
\end{equation}

Then it follows from (17) and (18) that h satisfies the assumptions in Lemma (4.3), and thus, h is a constant and

\[ F_D^{\alpha,q}(f)(x) = Ce^{-\frac{k^2}{4\pi}x^2}. \]

Since \(ab > \frac{k^2}{4}\), (15) holds whenever \(C = 0\) and the Plancherel formula implies that \(f = 0\) almost everywhere.

**Case 2: \((ab = \frac{k^2}{4})\)**

As in the previous case, we have the relation (17) and:

\begin{align*}
\int_{-\infty}^{+\infty} \log^+ \left| \frac{h(x)}{\lambda} \right||x|^{2\alpha+1}dx &= \int_{-\infty}^{+\infty} \log^+ e^{bx^2} F_D^{\alpha,q}(f)(x) \left| \left| \frac{F_D^{\alpha,q}(f)(x)}{\lambda} \right| \right| |x|^{2\alpha+1}dx \\
&< +\infty
\end{align*}

then h is a constant and we have

\[ F_D^{\alpha,q}(f)(x) = Ce^{-\frac{k^2}{4\pi}x^2} = Ce^{-bx^2}. \]

The relation (15) holds whenever \(|N| ≤ \lambda|\).
• Case 3: \( ab < \frac{k^2}{4} \)

Let \( a < t < \frac{k^2}{4b} \) by Theorem (4.5) we conclude that \( e^{ax^2}h_t^{\alpha,q}(x) \in L^{p_j}_{\alpha,q}(\mathbb{R}_q) \) for \( p_j \in [1, +\infty] \) and \( j = 1, 2 \). Then \( h_t^{\alpha,q} \) satisfy (14), on the other hand, we have:

\[
F_D^{\alpha,q}(h_t^{\alpha,q})(x) = e^{-\frac{k^2}{4t}x^2}
\]

then \( h_t^{\alpha,q} \) satisfy (15) because \( t < \frac{k^2}{4b} \).

\[
\blacksquare
\]

5 Applications

By the use of the theorem 4.6 we can easily deduce the \( L^{p_1} + L^{p_2} \)-version of Miyachi’s Theorem for the \( q^2 \)-analogue Fourier transform on the Real Line (see theorem 5.1) and \( L^{p_1} + L^{p_2} \)-version of Miyachi’s Theorem for the \( q \)-Bessel transform on the Real Line for the even functions (see corollary 5.2)

**Theorem 5.1** Let \( a, b > 0, p_1, p_2 \in [1, +\infty] \) and \( f \) be a function defined on \( \mathbb{R}_q \) such that:

\[
e^{ax^2}f \in L^{p_1}_{\alpha,q}(\mathbb{R}_q) + L^{p_2}_{\alpha,q}(\mathbb{R}_q)
\]

and

\[
\int_{-\infty}^{+\infty} \log^+ \left( \frac{\hat{f}(x; q^2)e^{bx^2}}{\lambda} \right)|x|^{2\alpha+1}dx < +\infty
\]

for some \( \lambda > 0 \), then

• if \( ab > \frac{k^2}{4} \) then \( f = 0 \) on \( \mathbb{R}_q \).

• if \( ab = \frac{k^2}{4} \) then \( f = Nh_a^{\alpha,q} \) where \( N \) a constant and \( |N| \leq \lambda \).

• if \( ab < \frac{k^2}{4} \) then there exists many functions.

**Proof:** We conclude from Theorem (4.6) for \( \alpha = -\frac{1}{2} \).  

**Corollary 5.2** Let \( a, b > 0, p_i \in [1, +\infty] \) where \( i = 1, \ldots, l \) and \( f \) be an even function defined on \( \mathbb{R}_q \) such that:

\[
e^{ax^2}f \in L^{p_1}_{\alpha,q}(\mathbb{R}_q) + L^{p_2}_{\alpha,q}(\mathbb{R}_q)
\]

(21)
and

\[
\int_{-\infty}^{+\infty} \log^+ \frac{\mathcal{F}_{\alpha,q}(f)(x)e^{bx^2}}{\lambda} |x|^{2\alpha+1} \, dx < +\infty
\]

(22)

for some \( \lambda > 0 \), then

- if \( ab > \frac{k^2}{4} \) then \( f = 0 \) on \( \mathbb{R}_q \).
- if \( ab = \frac{k^2}{4} \) then \( f = N h_{\alpha,q} \) where \( N \) a constant and \( |N| \leq \lambda \).
- if \( ab < \frac{k^2}{4} \) then there exists many functions.

**Proof:** For the even function we have \( \mathcal{F}_{\alpha,q}(f) = F_{\alpha,q}(f) \), then we conclude from Theorem (4.6).

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**References**


Miyachi’s Theorem for the $q$-Dunkl transform on the real line


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