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Seleberg Type Inequalities in Hilbert C*-Modules

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Abstract

In this paper we prove a type and refinement of Selberg type inequalities in Hilbert C*-modules. Some applications for improving the Cauchy-Schwartz, the Bessel, the Bombiari and The Boas - Bellman type inequality resulte are given.

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1 Introduction

The Selberg inequality. Let $y_1, \dots, y_n$, be nonzero vectors in a Hilbert space $\mathbb{X}$ with inner product $\langle , \rangle$. Then, for all $x \in \mathbb{X}$,

$$\sum_{j=1}^{n} \frac{|\langle x, y_j \rangle|^2}{\sum_{k=1}^{n} |\langle y_j, y_k \rangle|} \leq \|x\|^2 \tag{1}$$

In [7], the Selberg inequality is refined as follows: If $\langle y, y_i \rangle = 0$ for given $\{y_i\}$, then

$$|\langle x, y \rangle|^2 + \sum_{j=1}^{n} \frac{|\langle x, y_j \rangle|^2}{\sum_{k=1}^{n} |\langle y_j, y_k \rangle|} \|y\|^2 \leq \|x\|^2 \|y\|^2 \tag{2}$$

holds for all x in $\mathbb{X}$.

It might be useful to observe that, out of (1), one may get the following inequality

1. For $n = 1$ and $y = y_1$ the Cauchy-Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| \, . \tag{3}$$

2. For $y_1, \ldots, y_n$, be orthonormal sequence of vectors , the Bessel inequality

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \leq \|x\|^2 \tag{4}$$

3. The Bombieri inequality

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \leq \|x\|^2 \max_{1\leq j\leq n} \sum_{k=1}^{n} |\langle y_j, y_k\rangle| \tag{5}$$

4. The Boas - Bellman type inequality in [2]

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \leq \|x\|^2 \left(\max_{1\leq j\leq n} \|y_j\|^2 + (n-1) \max_{j\neq k} |\langle y_j, y_k\rangle| \right) \tag{6}$$

The goal of this paper is to show some related as well as a extension of the Selberg inequality to Hilbert C^*-module. We can obtain various particular inequalities in Hilbert C^*-module.

2 Preliminaries in Hilbert C^*-modules

In this section we briefly recall the definitions and examples of Hilbert C^*-modules. For information about Hilbert C^*-module, we refer to ([4,5,9]). Our references for C^*-algebras are ([1]).

Let $\mathbb{A}$ be a C^*-algebra (not necessarily unitary) and $\mathbb{X}$ be a complex linear space.

Definition 2.1. *A pre-Hilbert $\mathbb{A}$-module is a right $\mathbb{A}$-module $\mathbb{X}$ equipped with a sesquilinear map $\langle ., .\rangle : \mathbb{X} \times \mathbb{X} \to \mathbb{A}$ satisfying*

1. $\langle x, x\rangle \geq 0; \langle x, x\rangle = 0$ *if and only if* $x = 0$ *for all* $\mathbb{X}$ *in* $\mathbb{X}$,
2. $\langle x, \alpha y + \beta z\rangle = \alpha \langle x, y\rangle + \beta(\langle x, z\rangle)$ *for all* x, y, z *in* $\mathbb{X}, \alpha, \beta$ *in* $\mathbb{C}$,
3. $\langle x, y\rangle = \langle y, x\rangle^*$ *for all* x, y *in* $\mathbb{X}$,

4. $\langle x, ya\rangle = \langle x, y\rangle\, a$ *for all* x, y *in* $\mathbb{X}$*,* a *in* $\mathbb{A}$*.*

The map $\langle .,.\rangle$ *is called an* $\mathbb{A}$*-valued inner product of* $\mathbb{X}$*, and for* $x \in \mathbb{X}$*, we define* $||x|| = ||\langle x, x\rangle||^{\frac{1}{2}}$ *is a norm on* $\mathbb{X}$ *,where the latter norm denotes that in the* C^*-*algebra* $\mathbb{A}$ *. This norm makes* $\mathbb{X}$ *into a right normed module over* $\mathbb{A}$*. A pre-Hilbert module* $\mathbb{X}$ *is called a Hilbert* $\mathbb{A}-$ *module if it is complete with respect to its norm. Two typical examples of Hilbert* C^**-modules are as follows:*
(I) Every Hilbert space is a Hilbert C-module.
(II) Every C^* *algebra* $\mathbb{A}$ *is a Hilbert* $\mathbb{A}$ *-module via* $\langle a, b\rangle = a^*b (a, b \in \mathbb{A})$*.*
Notice that the inner product structure of a C^**-algebra is essentially more complicated than complex numbers.*

One may define an $\mathbb{A}-$*valued norm* $|.|$ *by* $|x| = \langle x, x\rangle^{\frac{1}{2}}$*. Clearly,* $\|x\| = \||x|\|$ *for each* $x \in \mathbb{X}$*. It is known that* $|.|$ *does not satisfy the triangle inequality in general.*

The following lemma is useful to prove this Selberg inequality.

Lemma 2.2. *(see [3]) Let* $\mathbb{A}$ *be a* C^*-*alegebra* $a, b, c \in \mathbb{A}$ *. Then*

$$a^*cb + b^*c^*a \le \|c\| \left(|a|^2 + |b|^2\right). \tag{7}$$

3 MAIN RESULT

We start our work by presenting a version of the Selberg inequality for Hilbert C^**-modules*

Theorem 3.1. *Let* $\mathbb{X}$ *be a Hilbert* $\mathbb{A}-$*module and* $y_1 \dots y_n$ *be a non zero vectors in* $\mathbb{X}$*. If* $x \in \mathbb{X}$ *then*

$$\sum_{j=1}^{n} \frac{|\langle y_j, x\rangle|^2}{\sum_{k=1}^{n} \|\langle y_j, y_k\rangle\|} \le |x|^2 \tag{8}$$

Proof.

$$\begin{aligned} 0 &\le \left| x - \sum_{j=1}^{n} y_j\alpha_j \right|^2 = \left\langle x - \sum_{j=1}^{n} y_j\alpha_j, x - \sum_{j=1}^{n} y_j\alpha_j \right\rangle \\ &= \langle x, x\rangle - \left\langle x, \sum_{j=1}^{n} y_j\alpha_j \right\rangle - \left\langle \sum_{j=1}^{n} y_j\alpha_j, x \right\rangle + \left\langle \sum_{j=1}^{n} y_j\alpha_j, \sum_{j=1}^{n} y_j\alpha_j \right\rangle \\ &= |x|^2 - \sum_{j=1}^{n} \alpha_j^* \langle y_j, x\rangle - \sum_{j=1}^{n} \langle x, y_j\rangle \alpha_j + \sum_{j,k=1}^{n} \alpha_j^* \langle y_j, y_k\rangle \alpha_k \\ &= |x|^2 - \sum_{j=1}^{n} \alpha_j^* \langle y_j, x\rangle - \sum_{j=1}^{n} \langle x, y_j\rangle \alpha_j + \frac{1}{2}\sum_{j,k=1}^{n} (\alpha_j^* \langle y_j, y_k\rangle \alpha_k + \alpha_k^* \langle y_k, y_j\rangle \alpha_j). \end{aligned}$$

By lemma (2.2) we have

$$\alpha_j^* \langle y_j, y_k \rangle \alpha_k + \alpha_k^* \langle y_k, y_j \rangle \alpha_j \leq |\alpha_j|^2 \|\langle y_j, y_k \rangle\| + |\alpha_k|^2 \|\langle y_j, y_k \rangle\|$$

then

$$\left| x - \sum_{j=1}^{n} y_j \alpha_j \right|^2 \leq |x|^2 - \sum_{j=1}^{n} \alpha_j^* \langle y_j, x \rangle - \sum_{j=1}^{n} \langle x, y_j \rangle \alpha_j + \frac{1}{2} \sum_{j,k=1}^{n} |\alpha_j|^2 \|\langle y_j, y_k \rangle\| + |\alpha_k|^2 \|\langle y_k, y_j \rangle\| \tag{9}$$

We choose

$$\alpha_j = \frac{\langle y_j, x \rangle}{\sum_{k=1}^{n} \|\langle y_j, y_k \rangle\|}$$

we get by using (9)

$$\begin{aligned} \left| x - \sum_{j=1}^{n} y_j \alpha_j \right|^2 &\leq |x|^2 - \sum_{j=1}^{n} \frac{|\langle y_j, x \rangle|^2}{\sum_{k=1}^{n} \|\langle y_j, y_k \rangle\|} - \sum_{j=1}^{n} \frac{|\langle y_j, x \rangle|^2}{\sum_{k=1}^{n} \|\langle y_j, y_k \rangle\|} \\ &+ \frac{1}{2} \sum_{j,k=1}^{n} \frac{|\langle y_j, x \rangle|^2 \|\langle y_j, y_k \rangle\|}{(\sum_{k=1}^{n} \|\langle y_j, y_k \rangle\|)^2} + \frac{1}{2} \sum_{j,k=1}^{n} \frac{|\langle y_j, x \rangle|^2 \|\langle y_j, y_k \rangle\|}{(\sum_{k=1}^{n} \|\langle y_j, y_k \rangle\|)^2} \\ &= |x|^2 - 2 \sum_{j=1}^{n} \frac{|\langle y_j, x \rangle|^2}{\sum_{k=1}^{n} \|\langle y_j, y_k \rangle\|} + \sum_{j=1}^{n} \frac{|\langle y_j, x \rangle|^2}{\sum_{k=1}^{n} \|\langle y_j, y_k \rangle\|} \\ &= |x|^2 - \sum_{j=1}^{n} \frac{|\langle y_j, x \rangle|^2}{\sum_{k=1}^{n} \|\langle y_j, y_k \rangle\|} \end{aligned}$$

Then

$$|x|^2 - \sum_{j=1}^{n} \frac{|\langle y_j, x \rangle|^2}{\sum_{k=1}^{n} \|\langle y_j, y_k \rangle\|} \geq 0. \tag{10}$$

Which complet the proof □

Corollary 3.2. *If $n = 1$ and $y_1 = y$ we have the Cauchy Schwartz inequality in Hilbert $C^* -$ module*

$$|\langle y, x \rangle|^2 \leq \|y\|^2 |x|^2 \tag{11}$$

With the following theorem we gave a refinement of Seleberg inequlity in Hilbert $C^* -$ module

Theorem 3.3. *Let $\mathbb{X}$ be a Hilbert $\mathbb{A}-$ module , y and $y_1 \dots y_n$ non zero vectors in $\mathbb{X}$ such that $\langle y, y_j \rangle = 0$ for $j = 1 \dots n$, if $x \in \mathbb{X}$ then*

$$|\langle y, x \rangle|^2 + \sum_{j=1}^{n} \frac{|\langle y_j, x \rangle|^2}{\sum_{k=1}^{n} \|\langle y_j, y_k \rangle\|} \|y\|^2 \leq |x|^2 \|y\|^2 \tag{12}$$

Proof. We put

$$u = x - \sum_{j=1}^{n} y_j \alpha_j$$

with

$$\alpha_j = \frac{\langle y_j, x\rangle}{\sum_{k=1}^{n} \|\langle y_j, y_k\rangle\|}.$$

We have from proof of theorem (3.1)

$$\begin{aligned} |u|^2 &= \left| x - \sum_{j=1}^{n} y_j \alpha_j \right|^2 \\ &\leq |x|^2 - \sum_{j=1}^{n} \frac{|\langle y_j, x\rangle|^2}{\sum_{k=1}^{n} \|\langle y_j, y_k\rangle\|}. \end{aligned}$$

Hence it follows that

$$\|y\|^2 \left(|x|^2 - \sum_{j=1}^{n} \frac{|\langle y_j, x\rangle|^2}{\sum_{k=1}^{n} \|\langle y_j, y_k\rangle\|} \right) \geq \|y\|^2 |u|^2 .$$

By using the Cauchy Shwartz inequality we have

$$\|y\|^2 |u|^2 \geq |\langle y, u\rangle|^2 .$$

But $\langle y, y_j\rangle = 0$ so

$$|\langle y, u\rangle|^2 = \left| \left\langle y, x - \sum_{j=1}^{n} y_j \alpha_j \right\rangle \right|^2 = |\langle y, x\rangle|^2 .$$

It follows that

$$\|y\|^2 \left(|x|^2 - \sum_{j=1}^{n} \frac{|\langle y_j, x\rangle|^2}{\sum_{k=1}^{n} \|\langle y_j, y_k\rangle\|} \right) \geq |\langle y, x\rangle|^2 . \tag{13}$$

Which completes the proof of Theorem . □

From Theorem (3.1) the following result of the Bessel , Bombieri and the Boas-Bellman type inequalities in Hilbert C^*-module can be obtained.

Corollary 3.4. *If $y_1, \ldots, y_n$ be a sequence of unit vectors in $\mathbb{X}$ such that $\langle y_j, y_k\rangle = 0$ for $1 \leq j \neq k \leq n$. Then*

$$\sum_{j=1}^{n} |\langle y_j, x\rangle|^2 \leq |x|^2 . \tag{14}$$

Proof. Clearly$\sum_{k=1}^{n} \|\langle y_j, y_k\rangle\| = 1$. Thus the result follows immediately from inequality (8). □

Corollary 3.5. *If $y_1, \ldots, y_n$ be a sequence vectors in $\mathbb{X}$ .If $x \in \mathbb{X}$ then*

$$\sum_{j=1}^{n} |\langle y_j, x\rangle|^2 \leq |x|^2 \max_{1\leq j\leq n} \sum_{k=1}^{n} \|\langle y_j, y_k\rangle\| . \tag{15}$$

Proof. We observe that

$$\sum_{k=1}^{n} \|\langle y_j, y_k\rangle\| \leq \max_{1\leq j\leq n} \sum_{k=1}^{n} \|\langle y_j, y_k\rangle\| .$$

We also have

$$\sum_{j=1}^{n} \frac{|\langle y_j, x\rangle|^2}{max_{1\leq j\leq n} \sum_{k=1}^{n} \|\langle y_j, y_k\rangle\|} \leq \sum_{j=1}^{n} \frac{|\langle y_j, x\rangle|^2}{\sum_{k=1}^{n} \|\langle y_j, y_k\rangle\|}.$$

Since

$$\sum_{j=1}^{n} \frac{|\langle y_j, x\rangle|^2}{max_{1\leq j\leq n} \sum_{k=1}^{n} \|\langle y_j, y_k\rangle\|} = \frac{\sum_{j=1}^{n} |\langle y_j, x\rangle|^2}{\max_{1\leq j\leq n} \sum_{k=1}^{n} \|\langle y_j, y_k\rangle\|}.$$

Then by using theorem (3.1) we get

$$\frac{\sum_{j=1}^{n} |\langle y_j, x\rangle|^2}{\max_{1\leq j\leq n} \sum_{k=1}^{n} \|\langle y_j, y_k\rangle\|} \leq |x|^2 .$$

Which completes the ptoof of Corollary □

Corollary 3.6. *If $y_1, \ldots, y_n$ be a sequence vectors in $\mathbb{X}$ .If $x \in \mathbb{X}$ then*

$$\sum_{j=1}^{n} |\langle y_j, x\rangle|^2 \leq |x|^2 \left(\max_{1\leq j\leq n} \|y_j\|^2 + (n-1) \max_{k\neq j} \|\langle y_j, y_k\rangle\| \right) . \tag{16}$$

Proof. Observe that,

$$\sum_{k=1}^{n} \|\langle y_j, y_k\rangle\| \leq \max_{1\leq j\leq n} \|y_j\|^2 + (n-1) \max_{j\neq k} \|\langle y_j, y_k\rangle\| .$$

We also have

$$\sum_{j=1}^{n} \frac{|\langle y_j, x\rangle|^2}{\max_{1\leq j\leq n} \|y_j\|^2 + (n-1) \max_{j\neq k} \|\langle y_j, y_k\rangle\|} \leq \sum_{j=1}^{n} \frac{|\langle y_j, x\rangle|^2}{\sum_{k=1}^{n} \|\langle y_j, y_k\rangle\|}.$$

Then by using the Selberg inequality (8) we get the result . □

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