∂-Normal and rg-Normal Fuzzy Biclosure Spaces

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Abstract

The purpose of this paper is to introduce the concept of ∂-normal fuzzy biclosure spaces and rg-normal fuzzy biclosure spaces and investigate some of their properties.

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1 Introduction

In [8], Mashhour and Ghanim have introduced the concept of fuzzy closure spaces as extension of Čech closure spaces [6]. They have extended the notions of subspaces, sums and products of fuzzy closure spaces. Later, Srivastava and Srivastava [9] have studied subspaces, sums, and products of the modification fuzzy closure spaces. Moreover, they have studied some separation axioms in such spaces. In [7], Elsalamony has introduced the concept of weak fuzzy closure spaces and studied some of their properties. In 2008, Boonpok and Khampakdee [5] have introduced and studied two notions of closed sets in closure spaces. Later, Boonpok [1] has introduced the notion of biclosure spaces. Furthermore, he has studied normal biclosure spaces [2], generalized normal biclosure spaces [3] and ∂-normal biclosure spaces [4] by using the concepts of closed sets, g-closed sets and ∂-closed sets, respectively. In [10], Tapi and Navalakhe have defined and studied the concept of fuzzy biclosure spaces. In this paper, we introduce the notions of regular fuzzy closed sets and
rg-fuzzy closed sets in a fuzzy closure spaces. Next, we introduce the concepts of \( \partial \)-normal fuzzy biclosure spaces and rg-normal fuzzy biclosure spaces and investigate some of their properties.

2 Preliminaries

First, we recall some basic definitions and notations of fuzzy sets. Let \( X \) be a nonempty set and \( I = [0,1] \). The set of all functions from \( X \) into \( I \) is denoted by \( I^X \). A member of \( I^X \) is called a fuzzy set in \( X \). The support of a fuzzy set \( \mu \) is denoted by \( \text{supp}(\mu) := \{ x \in X : \mu(x) > 0 \} \). Let \( \mu \) and \( \nu \) be fuzzy sets in \( X \). We say that \( \mu \) contained in \( \nu \), denoted by \( \mu \leq \nu \), if \( \mu(x) \leq \nu(x) \) for all \( x \in X \). It is clear that \( \mu \) and \( \nu \) are equal (i.e., \( \mu = \nu \)) if and only if \( \mu \leq \nu \) and \( \nu \leq \mu \). The union of \( \mu \) and \( \nu \), denoted by \( \mu \lor \nu \), is defined as the fuzzy set that \( (\mu \lor \nu)(x) = \max\{\mu(x),\nu(x)\} \) for all \( x \in X \). The intersection of \( \mu \) and \( \nu \), denoted by \( \mu \land \nu \), is defined as the fuzzy set that \( (\mu \land \nu)(x) = \min\{\mu(x),\nu(x)\} \) for all \( x \in X \). The complement of \( \mu \), denoted by \( 1 - \mu \), is defined as the fuzzy set that \( (1 - \mu)(x) = 1 - \mu(x) \) for every \( x \in X \).

More generally, for a family \( \{\mu_i : i \in J\} \) of fuzzy sets in \( X \), the union \( \bigvee_{i \in J} \mu_i \) and intersection \( \bigwedge_{i \in J} \mu_i \) are defined by \( \bigvee_{i \in J} \mu_i(x) = \sup\{\mu_i(x) : i \in J\} \) for all \( x \in X \) and \( \bigwedge_{i \in J} \mu_i(x) = \inf\{\mu_i(x) : i \in J\} \) for all \( x \in X \), respectively. If \( x \in X \) and \( r \in (0,1] \), by the fuzzy point \( x_r \) we mean the fuzzy set in \( X \) which takes the value \( r \) at the point \( x \) and 0 elsewhere. We say that a fuzzy point \( x_r \) contained in a fuzzy set \( \mu \), denoted by \( x_r \in \mu \), if \( r \leq \mu(x) \). If \( Y \subseteq X \), \( 1_Y \) denotes the characteristic function of \( Y \). Any fuzzy set \( \mu \) in \( Y \subseteq X \) will be identified with the fuzzy set in \( X \) (having the same notation as \( \mu \)) which takes the same values as \( \mu \) for \( x \in Y \) and 0 for \( x \in X - Y \). If \( Y \subseteq X \), any fuzzy set \( \mu \) in \( X \) which takes 0 for all \( x \in X - Y \) can be identified with the fuzzy set in \( Y \) which takes the same values as \( \mu \) for \( x \in Y \) (having the same notation as \( \mu \)). The null fuzzy set \( 0_X \) in \( X \) is defined as \( 0_X(x) = 0 \) for all \( x \in X \), and the whole fuzzy set in \( X \) is \( 1_X \). Let \( X \) and \( Y \) be nonempty sets and let \( f : X \rightarrow Y \) be a function. For a fuzzy set \( \nu \) in \( Y \), the inverse image of \( \nu \) under \( f \) is the fuzzy set \( f^{-1}(\nu) \) in \( X \) by the rule \( f^{-1}(\nu)(x) = \nu(f(x)) \) for all \( x \in X \). Conversely, for a fuzzy set \( \mu \) in \( X \), the image of under \( f \) is the fuzzy set \( f(\mu) \) in \( Y \) defined, for \( y \in Y \), by the rule

\[
f(\mu)(y) = \begin{cases} \sup\{\mu(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \text{ is nonempty,} \\ 0 & \text{otherwise,} \end{cases}
\]

where \( f^{-1}(y) = \{ x : f(x) = y \} \). If \( f : X \rightarrow Y \), then we have the following properties for image and inverse image fuzzy sets under \( f \) where \( \eta, \mu, \mu_i \in I^X \) and \( \gamma, \nu, \nu_i \in I^Y \) for all \( i \in J \).

1. \( \nu \geq f(f^{-1}(\nu)) \) with equality if \( f \) is surjective.
(2) $\mu \leq f^{-1}(f(\mu))$ with equality if $f$ is injective.

(3) $f^{-1}(1 - \nu) = 1 - f^{-1}(\nu)$.

(4) $f(1 - \mu) = 1 - f(\mu)$ if $f$ is bijective.

(5) $f^{-1}(\bigvee_{i \in J} \nu_i) = \bigvee_{i \in J} f^{-1}(\nu_i)$.

(6) $f^{-1}(\bigwedge_{i \in J} \nu_i) = \bigwedge_{i \in J} f^{-1}(\nu_i)$.

(7) If $\gamma \leq \nu$, then $f^{-1}(\gamma) \leq f^{-1}(\nu)$.

(8) If $\eta \leq \mu$, then $f(\eta) \leq f(\mu)$.

(9) $f(\bigvee_{i \in J} \mu_i) = \bigvee_{i \in J} f(\mu_i)$.

(10) $f(\bigwedge_{i \in J} \mu_i) = \bigwedge_{i \in J} f(\mu_i)$ with equality if $f$ is injective.

Next, we recall some basic definitions and notations of fuzzy closure spaces.

**Definition 2.1.** [7] Let $X$ be a nonempty set. A function $u : I^X \to I^X$ defined on the family $I^X$ of all fuzzy sets in $X$ is called a fuzzy closure operator on $X$ if the following conditions are satisfied

(1) $u(0_X) = 0_X$,

(2) $\mu \leq u(\mu)$ for all $\mu \in I^X$,

(3) if $\mu \leq \nu$, then $u(\mu) \leq u(\nu)$ for all $\mu, \nu \in I^X$.

And the pair $(X, u)$ is called a fuzzy closure space. When the context is clear, we shall write $u\mu$ for $u(\mu)$.

**Definition 2.2.** Let $X$ be a nonempty set. A fuzzy operator $u : I^X \to I^X$ on $X$ is called additive (respectively, idempotent) if $u(\mu \lor \nu) = u\mu \lor u\nu$ for all $\mu, \nu \in I^X$ (respectively, $uu\mu = u\mu$ for all $\mu \in I^X$).

**Definition 2.3.** [7] A fuzzy subset $\mu$ of a fuzzy closure space $(X, u)$ is said to be fuzzy closed if $u\mu = \mu$ and it is fuzzy open if its complement is fuzzy closed.

**Remark 2.4.** The null fuzzy set and the whole fuzzy set are both fuzzy open and fuzzy closed.

**Definition 2.5.** [10] A fuzzy closure space $(Y, v)$ is said to be a fuzzy subspace of $(X, u)$ if $Y \subseteq X$ and $v(\mu) = u(\mu) \land 1_Y$ for each fuzzy subset $\mu \in I^Y$. If $1_Y$ is fuzzy closed in $(X, u)$, then the fuzzy subspace $(Y, v)$ of $(X, u)$ is also said to be fuzzy closed.
Definition 2.6. [10] A fuzzy biclosure space is a triple $(X, u_1, u_2)$ where $X$ is a set and $u_1$, $u_2$ are two fuzzy closure operators on $X$.

Definition 2.7. [10] Let $(X, u_1, u_2)$ be a fuzzy biclosure space. A fuzzy biclosure space $(Y, v_1, v_2)$ is called a fuzzy subspace of $(X, u_1, u_2)$ if $Y \subseteq X$ and $v_i \mu = u_i \mu \wedge 1_Y$ for each $i \in \{1, 2\}$ and for all $\mu \in I^X$. If $1_Y$ is fuzzy closed in $(X, u_1)$ and $(X, u_2)$, then the fuzzy subspace $(Y, v_1, v_2)$ of $(X, u_1, u_2)$ is also said to be fuzzy closed.

Now, we give the concepts of g-fuzzy closed sets and $\partial$-fuzzy closed sets in closure spaces and discuss some of their properties which will be useful in our work. We begin our discussion by first studying the properties of fuzzy closed and fuzzy open set.

Proposition 2.8. Let $(X, u)$ be a fuzzy closure space. If $\mu_i$ is fuzzy closed in $(X, u)$ for all $i \in J$, then $\bigwedge_{i \in J} \mu_i$ is fuzzy closed in $(X, u)$.

Proposition 2.9. Let $(X, u)$ be a fuzzy closure space. If $\mu_i$ is fuzzy open in $(X, u)$ for all $i \in J$, then $\bigvee_{i \in J} \mu_i$ is fuzzy open in $(X, u)$.

Theorem 2.10. Let $(X, u)$ be a fuzzy closure space and let $(Y, v)$ be a fuzzy closed subspace of $(X, u)$ and $\mu$ a fuzzy set in $Y$. Then $\mu$ is fuzzy closed in $(Y, v)$ if and only if there exists a fuzzy closed set $\nu$ in $(X, u)$ such that $\mu = \nu \wedge 1_Y$.

Theorem 2.11. Let $(X, u)$ be a fuzzy closure space and let $(Y, v)$ be a fuzzy closed subspace of $(X, u)$ and $\mu$ a fuzzy set in $Y$. Then $\mu$ is fuzzy open in $(Y, v)$ if and only if there exists a fuzzy open set $\nu$ in $(X, u)$ such that $\mu = \nu \wedge 1_Y$.

Definition 2.12. Let $(X, u)$ be a fuzzy closure space. A fuzzy set $\mu$ in $(X, u)$ is called generalized fuzzy closed, briefly g-fuzzy closed, if $u \mu \leq \nu$ whenever $\nu$ is a fuzzy open set in $(X, u)$ with $\mu \leq \nu$. A fuzzy set $\mu$ in $(X, u)$ is called generalized fuzzy open, briefly g-fuzzy open, if its complement is g-fuzzy closed.

Remark 2.13. In a fuzzy closure space, every fuzzy closes set is a g-fuzzy closed set. The converse is not true as can be seen from the following example.

Example 2.14. Let $X = \{a, b\}$ and define a fuzzy closure operator $u$ on $X$ by

$$ u \mu = \begin{cases} 0_X, & \text{if } \mu = 0_X, \\ 1_X, & \text{otherwise} \end{cases} $$

for each $\mu \in I^X$. Then $1_{\{a\}}$ is g-fuzzy closed set but it is not fuzzy closed.

Proposition 2.15. Let $(X, u)$ be a fuzzy closure space and $Y$ a subset of $X$. If $\mu$ is a g-fuzzy closed set in $(X, u)$ and $1_Y$ is a fuzzy closed set in $(X, u)$, then $\mu \wedge 1_Y$ is a g-fuzzy closed set in $(X, u)$.
Proof. Let \( \nu \) be a fuzzy open set in \((X, u)\) such that \( \mu \wedge 1_Y \leq \nu \). Then \( \mu \leq \nu \vee (1 - 1_Y) \). Since \( 1_Y \) is fuzzy closed in \((X, u)\), \( 1 - 1_Y \) is fuzzy open in \((X, u)\). Thus, by Proposition 2.9, \( \nu \vee (1 - 1_Y) \) is fuzzy open in \((X, u)\). Since \( \mu \) is a g-fuzzy closed set in \((X, u)\), \( u \mu \leq \nu \vee (1 - 1_Y) \). Consequently, \( u \mu \wedge 1_Y \leq \nu \). Since \( 1_Y \) is fuzzy closed in \((X, u)\), \( u(\mu \wedge 1_Y) \leq u \mu \wedge 1_Y \leq \nu \). Hence, \( \mu \wedge 1_Y \) is g-fuzzy closed in \((X, u)\).

Proposition 2.16. Let \((X, u)\) be a fuzzy closure space and \( Y \) a subset of \( X \). If \( \mu \) is a g-fuzzy open set in \((X, u)\) and \( 1_Y \) is a fuzzy open set in \((X, u)\), then \( \mu \vee 1_Y \) is a g-fuzzy open set in \((X, u)\).

Proof. Assume that \( \mu \) is a g-fuzzy open set in \((X, u)\) and \( 1_Y \) is a fuzzy open set in \((X, u)\). Then \( 1 - \mu \) is a g-fuzzy closed set in \((X, u)\) and \( 1_X - \nu = 1 - 1_Y \) is a fuzzy closed set in \((X, u)\). By Proposition 2.15, \( (1 - \mu) \vee (1 - 1_Y) \) is a g-fuzzy closed set in \((X, u)\). Hence \( \mu \vee 1_Y = 1 - ((1 - \mu) \vee (1 - 1_Y)) \) is a g-fuzzy open set in \((X, u)\).

Lemma 2.17. Let \((X, u)\) be a fuzzy closure space and let \((Y, v)\) be a fuzzy closed subspace of \((X, u)\). If \( \mu \) is a g-fuzzy closed set in \((X, u)\), then \( \mu \wedge 1_Y \) is a g-fuzzy closed set in \((Y, v)\).

Proof. Let \( \eta \) be a fuzzy open set in \((Y, u)\) such that \( \mu \wedge 1_Y \leq \eta \). Then, by Theorem 2.11, there exists a fuzzy open set \( \nu \) in \((X, u)\) such that \( \eta = \nu \wedge 1_Y \), and so \( \mu \wedge 1_Y \leq \nu \wedge 1_Y \). Thus \( \mu \leq \nu \vee (1 - 1_Y) \). Since \( \nu \) and \( 1 - 1_Y \) are fuzzy open sets in \((X, u)\), by Proposition 2.9, \( \nu \vee (1 - 1_Y) \) is a fuzzy open set in \((X, u)\). Since \( \mu \) is a g-fuzzy closed set in \((X, u)\), \( u \mu \leq \nu \vee (1 - 1_Y) \). Then \( u \mu \wedge 1_Y \leq \nu \wedge 1_Y = \eta \). Hence \( v(\mu \wedge 1_Y) = u(\mu \wedge 1_Y) \wedge 1_Y \leq u \mu \wedge 1_Y \leq \eta \). Therefore, \( \mu \wedge 1_Y \) is a g-fuzzy closed set in \((Y, v)\).

Lemma 2.18. Let \((X, u)\) be a fuzzy closure space and let \((Y, v)\) be a fuzzy closed subspace of \((X, u)\). If \( \mu \) is a g-fuzzy open set in \((X, u)\), then \( \mu \wedge 1_Y \) is a g-fuzzy open set in \((Y, v)\).

Proof. Assume that \( \mu \) is a g-fuzzy open set in \((X, u)\). Then \( 1 - \mu \) is a g-fuzzy closed set in \((X, u)\). By Lemma 2.17, \( (1 - \mu) \wedge 1_Y \) is a g-fuzzy closed set in \((Y, v)\). Since \( (1 - \mu) \wedge 1_Y = (1 - (\mu \wedge 1_Y)) \wedge 1_Y \), the complement of \( (1 - \mu) \wedge 1_Y \) in \( Y \) is \( \mu \wedge 1_Y \). Hence \( \mu \wedge 1_Y \) is a g-fuzzy open set in \((Y, v)\).

Definition 2.19. Let \((X, u)\) be a fuzzy closure space. A fuzzy set \( \mu \) in \((X, u)\) is called \( \partial \)-fuzzy closed if \( u \mu \leq \nu \) whenever \( \nu \) is an g-fuzzy open set in \((X, u)\) with \( \mu \leq \nu \). A fuzzy set \( \mu \) in \((X, u)\) is called \( \partial \)-fuzzy open if its complement is \( \partial \)-fuzzy closed.

Remark 2.20. For a fuzzy set \( \mu \) in a fuzzy closure space \((X, u)\), the following implications hold:

\[ \mu \text{ is fuzzy closed} \Rightarrow \mu \text{ is g-fuzzy closed} \Rightarrow \mu \text{ is } \partial \text{-fuzzy closed}. \]

None of these implications is reversible as shown by the following examples.
Example 2.21. Let $X = \{a, b, c, d\}$ and define a fuzzy closure operator $u$ on $X$ by

$$u\mu = \begin{cases} 0_X & \text{if } \mu = 0_X, \\ 1_{\{a,c\}} & \text{if } 0 < \mu \leq 1_{\{a\}}, \\ 1_{\{b,c\}} & \text{if } 0 < \mu \leq 1_{\{b\}}, \\ 1_{\{c,d\}} & \text{if } 0 < \mu \leq 1_{\{c,d\}}, \\ 1_X & \text{otherwise} \end{cases}$$

for each $\mu \in I^X$. Then $1_{\{a,c,d\}}$ is $\partial$-fuzzy closed set but it is not fuzzy closed.

Example 2.22. Let $X = \{c, d\}$ and define a fuzzy closure operator $u$ on $X$ by

$$u\mu = \begin{cases} 0_X & \text{if } \mu = 0_X, \\ 1_X & \text{otherwise} \end{cases}$$

for each $\mu \in I^X$. Then $1_{\{c\}}$ is $g$-fuzzy closed set but it is not $\partial$-fuzzy closed.

Definition 2.23. Let $(X, u)$ and $(Y, v)$ be fuzzy closure spaces. A map $f : (X, u) \to (Y, v)$ is called $\partial$-fuzzy irresolute if $f^{-1}(\nu)$ is a $\partial$-fuzzy closed set in $(X, u)$ for every $\partial$-fuzzy closed set $\nu$ in $(Y, v)$.

Clearly, a map $f : (X, u) \to (Y, v)$ is $\partial$-fuzzy irresolute if and only if $f^{-1}(\nu)$ is a $\partial$-fuzzy open set in $(X, u)$ for every $\partial$-fuzzy open set $\nu$ in $(Y, v)$.

Definition 2.24. Let $(X, u)$ and $(Y, v)$ be fuzzy closure spaces. A map $f : (X, u) \to (Y, v)$ is called preserve $\partial$-fuzzy closed if $f(\mu)$ is a $\partial$-fuzzy closed set in $(Y, v)$ for every $\partial$-fuzzy closed set $\mu$ in $(X, u)$.

3 $\partial$-normal fuzzy biclosure spaces

In this section, we introduce the notion $\partial$-normal fuzzy biclosure spaces and investigate some of their properties.

Definition 3.1. A fuzzy biclosure space $(X, u_1, u_2)$ is called a $\partial$-normal fuzzy biclosure space if any $\partial$-fuzzy closed set $\eta$ in $(X, u_1)$ and any $\partial$-fuzzy closed set $\gamma$ in $(X, u_2)$ with $\eta \cap \gamma = 0_X$, there exist a $\partial$-fuzzy open set $\mu$ in $(X, u_1)$ and a $\partial$-fuzzy open set $\nu$ in $(X, u_2)$ such that $\eta \leq \mu$, $\gamma \leq \nu$ and $\mu \cap \nu = 0_X$.

Example 3.2. Let $X = \{a, b\}$ and define two fuzzy closure operators $u_1$ and $u_2$ on $X$ by $u_1\mu = \mu$ and $u_2\mu = \mu$ for each $\mu \in I^X$. Then $(X, u_1, u_2)$ is a $\partial$-normal biclosure space.
Lemma 3.3. Let \((X, u)\) be a fuzzy closure space and let \((Y, v)\) be a fuzzy closed subspace of \((X, u)\). If \(\mu\) is a \(\partial\)-fuzzy closed set in \((X, u)\), then \(\mu \land 1_Y\) is a \(\partial\)-fuzzy closed set in \((Y, v)\).

Proof. Let \(\eta\) be a g-fuzzy open set in \((Y, v)\) such that \(\mu \land 1_Y \leq \eta\). Then \(\mu \leq \eta \lor (1 - 1_Y)\). Since \(1_Y\) is a fuzzy closed set in \((X, u)\), \(1_{X \setminus Y} = 1 - 1_Y\) is a fuzzy open set in \((X, u)\). By Lemma 2.16, \(\eta \lor (1 - 1_Y) = \eta \lor 1_{X \setminus Y}\) is a g-fuzzy open set in \((X, u)\). Since \(\mu\) is a \(\partial\)-fuzzy closed set in \((X, u)\), \(u\mu \leq \eta \lor (1 - 1_Y)\). Thus \(u\mu \land 1_Y \leq \eta\). Then \(v(\mu \land 1_Y) = u(\mu \land 1_Y) \land 1_Y \leq u\mu \land 1_Y \leq \eta\). Hence, \(\mu \land 1_Y\) is a \(\partial\)-fuzzy closed set in \((Y, v)\). \(\square\)

Lemma 3.4. Let \((X, u)\) be a fuzzy closure space and let \((Y, v)\) be a fuzzy closed subspace of \((X, u)\). If \(\mu\) is a \(\partial\)-fuzzy open set in \((X, u)\), then \(\mu \land 1_Y\) is a \(\partial\)-fuzzy open set in \((Y, v)\).

Proof. Assume that \(\mu\) is a \(\partial\)-fuzzy open set in \((X, u)\). Then \(1 - \mu\) is a \(\partial\)-fuzzy closed set in \((X, u)\). By Lemma 3.3, \((1 - \mu) \land 1_Y\) is a \(\partial\)-fuzzy closed set in \((Y, v)\). Since \((1 - \mu) \land 1_Y = (1 - (\mu \land 1_Y)) \land 1_Y\), the complement of \((1 - \mu) \land 1_Y\) in \(Y\) is \(\mu \land 1_Y\). Hence \(\mu \land 1_Y\) is a \(\partial\)-fuzzy open set in \((Y, v)\). \(\square\)

Lemma 3.5. Let \((X, u)\) be a fuzzy closure space and let \((Y, v)\) be a fuzzy closed subspace of \((X, u)\). If \(\mu\) is a \(\partial\)-fuzzy closed set in \((Y, v)\), then \(\mu\) is a \(\partial\)-fuzzy closed set in \((X, u)\).

Proof. Let \(\eta\) be a g-fuzzy open set in \((X, u)\) such that \(\mu \leq \eta\). Then, by Lemma 2.18, \(\eta \land 1_Y\) is a g-fuzzy open set in \((Y, v)\) such that \(\mu \leq \eta \land 1_Y\). Since \(\mu\) is a \(\partial\)-fuzzy closed set in \((Y, v)\), \(v\mu \leq \eta \land 1_Y\). Since \(1_Y\) is fuzzy closed in \((X, u)\) and \(\mu \leq 1_Y\), \(u\mu \leq 1_Y\), and so \(u\mu = u\mu \land 1_Y = v\mu \leq \eta \land 1_Y \leq \eta\). Then \(\mu\) is \(\partial\)-fuzzy closed in \((X, u)\). \(\square\)

Proposition 3.6. Let \((X, u_1, u_2)\) be a fuzzy biclosure space and let \((Y, v_1, v_2)\) be a fuzzy closed subspace of \((X, u_1, u_2)\). If \((X, u_1, u_2)\) is a \(\partial\)-normal fuzzy biclosure space, then \((Y, v_1, v_2)\) is a \(\partial\)-normal fuzzy biclosure space.

Proof. Let \(\eta\) be a \(\partial\)-fuzzy closed set in \((Y, v_1)\) and \(\gamma\) a \(\partial\)-fuzzy closed set in \((Y, v_2)\) such that \(\eta \land \gamma = 0_Y\). By Lemma 3.5, \(\eta\) is \(\partial\)-fuzzy closed in \((X, u_1)\) and \(\gamma\) is a \(\partial\)-fuzzy closed set in \((X, u_2)\). Moreover, \(\eta \land \gamma = 0_X\). Since \((X, u_1, u_2)\) is a \(\partial\)-normal fuzzy biclosure space, there exist a \(\partial\)-fuzzy open set \(\mu\) in \((X, u_1)\) and a \(\partial\)-fuzzy open set \(\nu\) in \((X, u_2)\) such that \(\eta \leq \mu\), \(\gamma \leq \nu\) and \(\mu \land \nu = 0_X\). Consequently, \(\eta \leq \mu \land 1_Y\), \(\gamma \leq \nu \land 1_Y\) and \((\mu \land 1_Y) \land (\nu \land 1_Y) = 0_Y\). By Lemma 3.4, \(\mu \land 1_Y\) is \(\partial\)-fuzzy open in \((Y, v_1)\) and \(\nu \land 1_Y\) is \(\partial\)-fuzzy open in \((Y, v_2)\). Therefore, \((Y, v_1, v_2)\) is a \(\partial\)-normal fuzzy biclosure space. \(\square\)

Definition 3.7. Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be a fuzzy biclosure spaces and let \(i \in \{1, 2\}\). A map \(f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)\) is called \(i\)-\(\partial\)-fuzzy irresolute
if the map $f : (X, u_i) \rightarrow (Y, v_i)$ is $\partial$-fuzzy irresolute. A map $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$ is called $\partial$-fuzzy irresolute if $f$ is $i\partial$-fuzzy irresolute for each $i \in \{1, 2\}$.

**Definition 3.8.** Let $(X, u_1, u_2)$ and $(Y, v_1, v_2)$ be a fuzzy biclosure spaces and let $i \in \{1, 2\}$. A map $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$ is called preserve $i\partial$-fuzzy closed if the map $f : (X, u_i) \rightarrow (Y, v_i)$ is preserve $\partial$-fuzzy closed. A map $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$ is called preserve $\partial$-fuzzy closed if $f$ is preserve $i\partial$-fuzzy closed for each $i \in \{1, 2\}$.

**Proposition 3.9.** Let $(X, u_1, u_2)$ and $(Y, v_1, v_2)$ be a fuzzy biclosure spaces and let $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$ be injective, preserve $\partial$-fuzzy closed and $\partial$-fuzzy irresolute. If $(Y, v_1, v_2)$ is a $\partial$-normal fuzzy biclosure space, then $(X, u_1, u_2)$ is a $\partial$-normal fuzzy biclosure space.

**Proof.** Let $\eta$ be a $\partial$-fuzzy closed set in $(X, u_1)$ and $\gamma$ a $\partial$-fuzzy closed set in $(X, u_2)$ such that $\eta \wedge \gamma = 0_X$. Since $f$ is injective and preserve $\partial$-fuzzy closed, $f(\eta)$ is a $\partial$-fuzzy closed set in $(Y, v_1)$ and $f(\gamma)$ is a $\partial$-fuzzy closed set in $(Y, v_2)$ such that $f(\eta) \wedge f(\gamma) = 0_Y$. Since $(Y, v_1, v_2)$ is a $\partial$-normal fuzzy biclosure space, there exist a $\partial$-fuzzy open set $\mu$ in $(Y, v_1)$ and a $\partial$-fuzzy open set $\nu$ in $(Y, v_2)$ such that $f(\eta) \leq \mu$, $f(\gamma) \leq \nu$ and $\mu \wedge \nu = 0_Y$. Since $f$ is $\partial$-fuzzy irresolute, $f^{-1}(\mu)$ is a $\partial$-fuzzy open set in $(X, u_1)$ and $\partial \cdot f^{-1}(\nu)$ is a $\partial$-fuzzy open set in $(X, u_2)$ such that $\eta \leq f^{-1}(\mu)$, $\gamma \leq f^{-1}(\nu)$ and $f^{-1}(\mu) \wedge f^{-1}(\nu) = 0_X$. Therefore, $(X, u_1, u_2)$ is a $\partial$-normal fuzzy biclosure space. $\square$

## 4 $rg$-normal fuzzy biclosure spaces

In this section, we give the notions of regular fuzzy closed sets and regular generalized fuzzy closed sets in fuzzy closure space. Next, we introduce the concept $rg$-normal fuzzy biclosure spaces and investigate some of their properties.

**Definition 4.1.** Let $(X, u)$ be a fuzzy closure space. A fuzzy set $\mu$ in $(X, u)$ is called regular fuzzy closed if $\mu$ is a fuzzy closed set in $(X, u)$ and $\mu = u(1 - u(1 - \mu))$. A fuzzy set $\mu$ in $(X, u)$ is called regular fuzzy open if its complement is regular fuzzy closed.

**Remark 4.2.** In a fuzzy closure space, every regular fuzzy closes set is a fuzzy closed set. The converse is not true as can be seen from the following example.

**Example 4.3.** Let $X = \{a, b\}$ and define a fuzzy closure operator $u$ on $X$ by 

$$u\mu = \begin{cases} 0_X, & \text{if } \mu = 0_X, \\ 1_{\{a\}}, & \text{if } 0 < \mu \leq 1_{\{a\}}, \\ 1_X, & \text{otherwise} \end{cases}$$

for each $\mu \in I^X$. Then $1_{\{a\}}$ is fuzzy closed set but it is not regular fuzzy closed.
Example 4.4. Let \( X = \{a, b\} \) and define a fuzzy closure operator \( u \) on \( X \) by

\[
u\mu = \begin{cases} 
0_X, & \text{if } \mu = 0_X, \\
1_{\{a, b\}}, & \text{if } 0 < \mu \leq 1_{\{a, b\}}, \\
1_{\{a, c\}}, & \text{if } 0 < \mu \leq 1_{\{c\}}, \\
1_X, & \text{otherwise}
\end{cases}
\]

for each \( \mu \in I^X \). Then \( 1_{\{a, b\}} \) is regular fuzzy closed set.

Proposition 4.5. Let \( (X, u) \) be a fuzzy closure space such that \( u \) is additive and let \( (Y, v) \) be a fuzzy closed subspace of \( (X, u) \) such that \( 1_Y \) is fuzzy open in \( (X, u) \). If \( \mu \) is a regular fuzzy closed set in \( (Y, v) \), then \( \mu \) is a regular fuzzy closed set in \( (X, u) \).

Proof. Assume that \( \mu \) is a regular fuzzy closed set in \( (Y, v) \). Since \( 1_Y \) is fuzzy closed in \( (X, u) \), \( u\mu \leq u1_Y = 1_Y \), and so \( u\mu = u\mu \wedge 1_Y = v\mu = \mu \). Then \( \mu \) is fuzzy closed in \( (X, u) \). Furthermore, \( 1 - v(1 - \mu) = 1 - (u(1 - \mu) \wedge 1_Y) = (1 - u(1 - \mu)) \vee (1 - 1_Y) \). Since \( u \) is additive and \( 1_Y \) is fuzzy open in \( (X, u) \), \( u(1 - v(1 - \mu)) = u(1 - u(1 - \mu)) \vee u(1 - 1_Y) = u(1 - u(1 - \mu)) \vee (1 - 1_Y) \). This implies

\[
\mu = v(1 - v(1 - \mu)) \\
= u(1 - v(1 - \mu)) \wedge 1_Y \\
= (u(1 - u(1 - \mu)) \vee (1 - 1_Y)) \wedge 1_Y \\
= (u(1 - u(1 - \mu)) \wedge 1_Y) \vee ((1 - 1_Y) \wedge 1_Y) \\
= u(1 - u(1 - \mu)) \wedge 1_Y \\
\leq u(1 - u(1 - \mu))
\]

But \( \mu \geq u(1 - u(1 - \mu)) \), \( \mu = u(1 - u(1 - \mu)) \). Hence \( \mu \) is regular fuzzy closed in \( (X, u) \). \( \square \)

Definition 4.6. Let \( (X, u) \) and \( (Y, v) \) be fuzzy closure spaces. A map \( f : (X, u) \to (Y, v) \) is called preserve regular fuzzy closed if \( f(\mu) \) is a regular fuzzy closed set in \( (Y, v) \) for every regular fuzzy closed set \( \mu \) in \( (X, u) \).

Definition 4.7. Let \( (X, u) \) be a fuzzy closure space. A fuzzy set \( \mu \) in \( (X, u) \) is called regular generalized fuzzy closed, briefly rg-fuzzy closed, if \( u\mu \leq \nu \) whenever \( \nu \) is a regular fuzzy open set in \( (X, u) \) with \( \mu \leq \nu \). A fuzzy set \( \mu \) in \( (X, u) \) is called regular generalized fuzzy open, briefly rg-fuzzy open, if its complement is rg-fuzzy closed.

Remark 4.8. In a fuzzy closure space, every g-fuzzy closes set is a rg-fuzzy closed set. The converse is not true as can be seen from the following example.
Example 4.9. Let $X = \{a, b\}$ and define a fuzzy closure operator $u$ on $X$ by

$$u\mu = \begin{cases} 0_X, & \text{if } \mu = 0_X, \\ 1_{\{a\}}, & \text{if } 0 < \mu \leq 1_{\{a\}}, \\ 1_X, & \text{otherwise} \end{cases}$$

for each $\mu \in I^X$. Then $1_{\{b\}}$ is rg-fuzzy closed set but it is not g-fuzzy closed.

Definition 4.10. Let $(X, u)$ and $(Y, v)$ be fuzzy closure spaces. A map $f : (X, u) \rightarrow (Y, v)$ is called rg-fuzzy irresolute if $f^{-1}(\nu)$ is a rg-fuzzy closed set in $(X, u)$ for every rg-fuzzy closed set $\nu$ in $(Y, v)$.

Clearly, a map $f : (X, u) \rightarrow (Y, v)$ is rg-fuzzy irresolute if and only if $f^{-1}(\nu)$ is a rg-fuzzy open set in $(X, u)$ for every rg-fuzzy open set $\nu$ in $(Y, v)$.

Definition 4.11. A fuzzy biclosure space $(X, u_1, u_2)$ is said to be a rg-normal fuzzy biclosure space, if any regular fuzzy closed set $\eta$ in $(X, u_1)$ and any regular fuzzy closed set $\gamma$ in $(X, u_2)$ with $\eta \wedge \gamma = 0_X$, there exist a rg-fuzzy open set $\mu$ in $(X, u_1)$ and a rg-fuzzy open set $\nu$ in $(X, u_2)$ such that $\eta \leq \mu$, $\gamma \leq \nu$ and $\mu \wedge \nu = 0_X$.

Remark 4.12. Every $\partial$-normal fuzzy biclosure space is a rg-normal fuzzy biclosure space. The converse is not true as can be seen from the following example.

Example 4.13. Let $X = \{a, b, c\}$ and define two closure operators $u_1$ and $u_2$ on $X$ by

$$u_1\mu = u_2\mu = \begin{cases} 0_X, & \text{if } \mu = 0_X, \\ 1_{\{a\}}, & \text{if } 0 < \mu \leq 1_{\{a\}}, \\ 1_{\{b\}}, & \text{if } 0 < \mu \leq 1_{\{b\}}, \\ 1_X, & \text{otherwise} \end{cases}$$

for each $\mu \in I^X$. Then $(X, u_1, u_2)$ is a rg-normal fuzzy biclosure space but it is not a $\partial$-normal fuzzy biclosure space.

Definition 4.14. A fuzzy closure space $(X, u)$ is said to be a $T^*_\mathcal{F}$-space if every rg-fuzzy closed set in $(X, u)$ is fuzzy closed in $(X, u)$.

Proposition 4.15. Let $(X, u_1, u_2)$ be a fuzzy biclosure space such that $(X, u_i)$ is $T^*_\mathcal{F}$-space and $u_i$ is additive for all $i \in \{1, 2\}$ and let $(Y, v_1, v_2)$ be a fuzzy closed subspace of $(X, u_1, u_2)$ such that $1_Y$ is fuzzy open in $(X, u_1)$ and $(X, u_2)$. If $(X, u_1, u_2)$ is a rg-normal fuzzy biclosure space, then $(Y, v_1, v_2)$ is a rg-normal fuzzy biclosure space.
Proof. Let $\eta$ be a regular fuzzy closed set in $(Y, v_1)$ and $\gamma$ a regular fuzzy closed set in $(Y, v_2)$ such that $\eta \wedge \gamma = 0_Y$. By Proposition 4.5, $\eta$ is regular fuzzy closed in $(X, u_1)$ and $\gamma$ is a regular fuzzy closed set in $(X, u_2)$. Moreover, $\eta \wedge \gamma = 0_X$. Since $(X, u_1, u_2)$ is a rg-normal fuzzy biclosure space, there exist a rg-fuzzy open set $\mu$ in $(X, u_1)$ and a rg-fuzzy open set $\nu$ in $(X, u_2)$ such that $\eta \leq \mu$, $\gamma \leq \nu$ and $\mu \wedge \nu = 0_X$. Consequently, $\eta \leq \mu \wedge 1_Y$, $\gamma \leq \nu \wedge 1_Y$ and $(\mu \wedge 1_Y) \wedge (\nu \wedge 1_Y) = 0_Y$. Since $(X, u_1)$ and $(X, u_2)$ is a $T_i^*$-space, we obtain that $\mu$ and $\nu$ are fuzzy open in $(X, u_1)$ and $(X, u_2)$, respectively. Thus $\mu \wedge 1_Y$ and $\nu \wedge 1_Y$ are fuzzy open in $(Y, v_1)$ and $(Y, v_2)$, respectively. Hence $\mu \wedge 1_Y$ is rg-fuzzy open in $(Y, v_1)$ and $\nu \wedge 1_Y$ is rg-fuzzy open in $(Y, v_2)$. Therefore, $(Y, v_1, v_2)$ is a rg-normal fuzzy biclosure space.

**Definition 4.16.** Let $(X, u_1, u_2)$ and $(Y, v_1, v_2)$ be a fuzzy biclosure spaces and let $i \in \{1, 2\}$. A map $f : (X, u_i, u_2) \to (Y, v_1, v_2)$ is said to be $i$-rg-fuzzy irresolute if the map $f : (X, u_i) \to (Y, v_i)$ is rg-fuzzy irresolute. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called rg-fuzzy irresolute if $f$ is $i$-rg-fuzzy irresolute for each $i \in \{1, 2\}$.

**Definition 4.17.** Let $(X, u_1, u_2)$ and $(Y, v_1, v_2)$ be a fuzzy biclosure spaces and let $i \in \{1, 2\}$. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called preserve $i$-regular fuzzy closed if the map $f : (X, u_i) \to (Y, v_i)$ is preserve regular fuzzy closed. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called preserve regular fuzzy closed if $f$ is preserve $i$-regular fuzzy closed for each $i \in \{1, 2\}$.

**Proposition 4.18.** Let $(X, u_1, u_2)$ and $(Y, v_1, v_2)$ be a fuzzy biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be injective, preserve regular fuzzy closed and rg-fuzzy irresolute. If $(Y, v_1, v_2)$ is a rg-normal fuzzy biclosure space, then $(X, u_1, u_2)$ is a rg-normal fuzzy biclosure space.

**Proof.** Let $\eta$ be a regular fuzzy closed set in $(X, u_1)$ and $\gamma$ a regular fuzzy closed set in $(X, u_2)$ such that $\eta \wedge \gamma = 0_X$. Since $f$ is injective and preserve regular fuzzy closed, $f(\eta)$ is a regular fuzzy closed set in $(Y, v_1)$ and $f(\gamma)$ is a regular fuzzy closed set in $(Y, v_2)$ such that $f(\eta) \wedge f(\gamma) = 0_Y$. Since $(Y, v_1, v_2)$ is a rg-normal fuzzy biclosure space, there exist a rg-fuzzy open set $\mu$ in $(Y, v_1)$ and a rg-fuzzy open set $\nu$ in $(Y, v_2)$ such that $\eta \leq f(\mu)$, $\gamma \leq f(\nu)$ and $\mu \wedge \nu = 0_Y$. Since $f$ is rg-fuzzy irresolute, $f^{-1}(\mu)$ is a rg-fuzzy open set in $(X, u_1)$ and $f^{-1}(\nu)$ is a rg-fuzzy open set in $(X, u_2)$ such that $\eta \leq f^{-1}(\mu)$, $\gamma \leq f^{-1}(\nu)$ and $f^{-1}(\mu) \wedge f^{-1}(\nu) = 0_X$. Therefore, $(X, u_1, u_2)$ is a rg-normal fuzzy biclosure space.

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