Ion-Acoustic Solitary Waves in an Inhomogeneous Magnetized Two-Ion-Temperature Dusty Plasma

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Abstract. This paper investigates the variable-coefficient Zakharov-Kuznetsov equation which governs the two-dimensional ion-acoustic waves obliquely propagating in an inhomogeneous magnetized two-ion-temperature dusty plasma. Via a simplified bilinear method, the N-soliton solution is constructed. The analysis confirms the fact that certain equations which have N-soliton solutions, have simultaneously, N-singular soliton solutions.

Keywords: N-soliton solution; variable-coefficient Zakharov-Kuznetsov equation; magnetized two-ion-temperature dusty plasma

Introduction

It is known that the Zakharov–Kuznetsov (ZK) equation is a two-dimensional generalization of the Korteweg-de Vries (KdV) equation, which has been initially derived for describing the weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma comprised of the cold ions and hot isothermal electrons. The ZK equation is given as follows [14]

\[ u_t + \delta u_x + \alpha uu_x + \beta u_{xxx} + \gamma u_{xyy} = 0, \]

(1)

and it is also a model for the vortices in geophysical flows since it supports the stable lump solitary waves. Moreover, by employing the multi-dimensional reductive perturbation technique Eq. (1) has been derived for the evolution of the electric potential perturbation [9].
In this paper, we will study the variable-coefficient Zakharov-Kuznetsov (vcZK) equation:

\[ u_t + \delta(t)u_x + \alpha(t)uu_x + \beta(t)u_{xxx} + \gamma(t)u_{xyy} = 0, \]  

where \( u(x, y, t) \) represents the electrostatic wave potential with the scaled “spatial” \( x, y \) and “temporal” \( t \), \( \delta(t) \) is an arbitrary function, and \( \alpha(t), \beta(t) \) and \( \gamma(t) \) are the coefficients of the nonlinear, dispersive and perturbed terms, respectively. The vcZK Eq. (2) governs the two-dimensional ion-acoustic waves obliquely propagating in an inhomogeneous magnetized two-ion-temperature dusty plasma. It is widely used in various branches of physics, such as plasma physics, fluid physics, and quantum field theory (See [9, 10, 11, 12, 13, 14] and the references therein). Much works have been done by many authors to study the vcZK Eq. (2) [9, 10, 11, 12, 13, 14].

The existence of solitons and soliton-like solutions for nonlinear PDFs is of particular significance because of their extensive applications in many physics areas such as nonlinear optics, plasmas, fluid mechanics, condensed matter, electromagnetics and many more. Envelope solitons are stable nonlinear wave packets that preserve their shape when propagating in a nonlinear dispersive medium. It is also of interest to note that the formation of these types of pulses is due to a perfect balance between the nonlinearity and dispersion effects.

In this paper, an efficient transformation method combined with the Hereman’s simplified method [1, 2, 3] will be used for a reliable treatment of the vcZK Eq. (2). We obtain new exact solutions including multiple soliton and multiple singular soliton solutions. These new solutions will help us understand and study the physical mechanism of this equation.

1. Soliton solutions for the vcZK equation

The Hirota bilinear method was first proposed by Hirota [4, 5, 6, 7, 8]. Now it is well-known as one of the most effective methods to construct multiple soliton solutions and also for classification of integrable equations. The Hirota’s direct method combined with the simplified version of Hereman and Zhuang [2, 3] has been applied successfully for the N-soliton solutions of integrable equations with constant coefficients. We improve and develop the approach to deal with the vcZK Eq. (2).
1.1. **Single soliton solution for the vcZK equation.** To obtain single soliton solutions, we first substitute

\[ u(x, y, t) = e^{k_i x + r_i y - \omega_i(t)} \]

into the linear terms of Eq. (2) and solve the resulting equation to determine the dispersion relation as

\[
\omega_i(t) = \int \left[ k_i \delta(t) + k_i^3 \beta(t) + k_i r_i^2 \gamma(t) \right] dt.
\]

This in turn gives the following phase variables

\[
\theta_i = k_i x + r_i y - \omega_i(t), \quad i = 1, 2, \ldots, N.
\]

We then substitute the single soliton solution

\[
u(x, y, t) = R (\ln f)_{xx}
\]

into Eq. (2) where the auxiliary function \( f(x, y, t) \) is given by

\[
f(x, y, t) = 1 + C_1 e^{\theta_1} = 1 + C_1 e^{k_1 x + r_1 y - \omega_1(t)},
\]

and solving for \( R \), we find that

\[
R = \frac{12 k_1^2 \beta(t) + r_1^2 \gamma(t)}{\alpha(t)}.
\]

In order to obtain a numerical value of \( R \), we impose the constraint conditions

\[
\alpha(t) = \lambda [\beta(t) + \gamma(t)], \quad k_i = r_i.
\]

where \( \lambda \) is a constant. Therefore, we will confirm that if the conditions (7) are satisfied, then Eq. (2) has \( N \)-soliton solutions.

This in turn gives the single soliton solution

\[
u(x, y, t) = 12 C_1 k_1^2 \frac{e^{\theta_1}}{(1 + C_1 e^{\theta_1})^2}
\]

where

\[
\theta_1 = k_1 x + k_1 y - \int \left[ k_1 \delta(t) + k_1^3 \alpha(t) \right] dt.
\]

For \( C_1 = 1 \), we obtain the single soliton solution

\[
u(x, y, t) = 12 k_1^2 \frac{e^{\theta_1}}{(1 + e^{\theta_1})^2} = 3 k_1^2 \sec h^2 \left( \frac{\theta_1}{2} \right)
\]
and for $C_1 = -1$ the single singular soliton solution

$$u(x, y, t) = -12k_1^2 \frac{e^{\theta_1}}{(1 - e^{\theta_1})^2} = -3k_1^2 \csc h^2(\frac{\theta_1}{2})$$

follows immediately.

For the two soliton solutions we set the auxiliary function

(10) $$f(x, y, t) = 1 + C_1 e^{\theta_1} + C_2 e^{\theta_2} + C_1 C_2 a_{12} e^{\theta_1 + \theta_2}.$$ 

Upon Substituting (5), with $f$ as in (10), into (2) and solving for the phase shift $a_{12}$, we obtain

(11) $$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

and hence we can generalize for other phase shifts by

(12) $$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leq i < j \leq 3.$$ 

The two soliton solutions are obtained by substituting (10) and (11) into

(13) $$u(x, y, t) = 12 (\ln f)_{xx},$$

where $C_1 = C_2 = 1$, and it is given in explicit form as

(14) $$u(x, y, t) = \frac{k_1^2 e^{\theta_1}(1 + a_{12} e^{2\theta_2}) + k_2^2 e^{\theta_2}(1 + a_{12} e^{2\theta_1}) + 2(k_1 - k_2)^2 e^{\theta_1 + \theta_2}}{(1 - e^{\theta_1 - e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}})^2},$$

where $\theta_1$ and $\theta_2$ are defined in (4).

However, the two singular soliton solutions are obtained by substituting (10) and (11) into (13) with $C_1 = C_2 = -1$,

(15) $$u(x, y, t) = -12 \frac{k_1^2 e^{\theta_1}(1 + a_{12} e^{2\theta_2}) + k_2^2 e^{\theta_2}(1 + a_{12} e^{2\theta_1}) - 2(k_1 - k_2)^2 e^{\theta_1 + \theta_2}}{(1 - e^{\theta_1 - e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}})^2}.$$ 

For the three-soliton solutions, we set the auxiliary function

$$f(x, y, t) = 1 + C_1 e^{\theta_1} + C_2 e^{\theta_2} + C_3 e^{\theta_3} + C_1 C_2 a_{12} e^{\theta_1 + \theta_2} + C_1 C_3 a_{13} e^{\theta_1 + \theta_3} + C_2 C_3 a_{23} e^{\theta_2 + \theta_3} + C_1 C_2 C_3 a_{123} e^{\theta_1 + \theta_2 + \theta_3},$$

(16) $$C_i = 1, \quad i = 1, 2, 3.$$ 

Substituting (16) and (13) into (2), we find that

$$a_{123} = a_{12} a_{13} a_{23}.$$
We emphasize the fact presented in [5, 6] that every solitonic equation that has generic $N = 3$ soliton solution has also soliton solutions for $N \geq 4$. This shows that Eq. (2) has $N$-soliton solutions which can be obtained for finite $N$, where $N > 1$. Generally, we can conjecture the $N$-soliton solutions for Eq. (2) as:

\[
\begin{align*}
  u(x, y, t) &= 12 (\ln f)_{xx}, \\
  f(x, y, t) &= \sum_{\mu = 0, 1} \exp \left( \sum_{1 \leq i < j} A_{ij} \mu_i \mu_j + \sum_{j=1}^{N} \mu_j \theta_j \right), \\
  \theta_j &= k_j x + k_j y - \int \left[ k_j \delta(t) + k_j^3 \alpha(t) \right] dt, \\
  \exp(A_{ij}) &= \frac{(k_i - k_j)^2}{(k_i + k_j)^2}.
\end{align*}
\]

Moreover, the three singular soliton solutions can be obtained in a similar manner for $C_1 = C_2 = C_3 = -1$ in (16).

2. Conclusion

Under consideration in this paper is the variable-coefficient Zakharov-Kuznetsov equation for the propagation of ion-acoustic waves in an inhomogeneous magnetized two-ion-temperature dusty plasma. In this paper, the simplified form of the Hirota bilinear method has been employed to derive the multi-soliton solutions and multiple singular soliton solutions. The solutions are formally obtained without any need to derive the bilinear forms. It is hoped that the proposed analysis will be useful for other models with inhomogeneities of media and nonuniformities of boundaries in the dynamics of ion-acoustic waves.

References


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