A Note on the Symmetric Properties
for the Tangent Polynomials

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Abstract

In [4], we studied the tangent numbers and polynomials. By using these numbers and polynomials, we give some interesting relations between the power sums and the tangent polynomials.

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1 Introduction

Tangent numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. In [4], we introduced the tangent numbers and polynomials. In this paper, by using the symmetry of $p$-adic integral on $\mathbb{Z}_p$, we give recurrence identities the tangent polynomials and the power sums.

Throughout this paper, let $p$ be a fixed odd prime number. The symbol, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. As well known definition, the $p$-adic absolute value is given by $|x|_p = p^{-r}$ where $x = p^r t^s$ with $(t, p) = (s, p) = (t, s) = 1$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. In this paper we assume
that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. We assume that $UD(\mathbb{Z}_p)$ is the space of the uniformly differentiable function on $\mathbb{Z}_p$. For $g \in UD(\mathbb{Z}_p)$, the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^{N-1}} f(x)(-1)^x,$$  

see [1, 2, 3, 4]. \hfill (1.1)

For $n \in \mathbb{N}$, let $g_n(x) = g(x + n)$ be translation. As well known equation, by (1.1), we have

$$\int_{\mathbb{Z}_p} g(x + n)d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l}g(l). \hfill (1.2)$$

In [4], we introduced the tangent numbers $T_n$ and polynomials $T_n(x)$ and investigate their properties. Let us define the tangent numbers $T_n$ and polynomials $T_n(x)$ as follows:

$$I_{-1}(e^{2yt}) = \int_{\mathbb{Z}_p} e^{2yt} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!}, \hfill (1.3)$$

$$I_{-1}(e^{(2y+x)t}) = \int_{\mathbb{Z}_p} e^{(2y+x)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}. \hfill (1.4)$$

The following elementary properties of the tangent numbers $T_n$ and polynomials $T_n(x)$ are readily derived form (1.3) and (1.4)(see, for details, [4]).

**Theorem 1.1** For any positive integer $n$, we have

$$\int_{\mathbb{Z}_p} (2x)^n d\mu_{-1}(x) = T_n, \quad \int_{\mathbb{Z}_p} (2y + x)^n d\mu_{-1}(y) = T_n(x).$$

**Theorem 1.2** For any positive integer $n$, we have

$$T_n(x) = \sum_{k=0}^{n} \binom{n}{k} T_k x^{n-k}.$$

2 The alternating sums of powers of consecutive even integers

By using (1.3), we give the alternating sums of powers of consecutive even integers as follows:

$$\sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \frac{2}{e^{2t} + 1} = 2 \sum_{n=0}^{\infty} (-1)^n e^{2nt}.$$
From the above, we obtain
\[ -\sum_{n=0}^{\infty}(-1)^ne^{(2n+2k)t} + \sum_{n=0}^{\infty}(-1)^{n-k}e^{2nt} = \sum_{n=0}^{k-1}(-1)^{n-k}e^{2nt}. \]

By using (1.3) and (1.4), we obtain
\[-\frac{1}{2}\sum_{j=0}^{\infty}T_j(2k)\frac{t^j}{j!} + \frac{1}{2}(-1)^{-k}\sum_{j=0}^{\infty}T_j\frac{t^j}{j!} = \sum_{j=0}^{\infty}\left((-1)^{-k}\sum_{n=0}^{k-1}(-1)^{n}(2n)^j\right)\frac{t^j}{j!}.\]

By comparing coefficients of \(\frac{t^j}{j!}\) in the above equation, we obtain
\[2\sum_{n=0}^{k-1}(-1)^{n}(2n)^j = (-1)^{k+1}T_j(2k) + T_j.\]

By using the above equation we arrive at the following theorem:

\textbf{Theorem 2.1} Let \(k\) be a positive integer. Then we obtain
\[T_j(k-1) = \sum_{n=0}^{k-1}(-1)^n(2n)^j = \frac{(-1)^{k+1}T_j(2k) + T_j}{2}.\]

3 The symmetry property of the deformed fermionic integral on \(\mathbb{Z}_p\)

In this section, we obtain recurrence identities the tangent polynomials and the alternating sums of powers of consecutive even integers. If \(n\) is odd from (1.2), we obtain
\[I_{-1}(g_n) + I_{-1}(g) = 2\sum_{k=0}^{n-1}(-1)^{n-1-k}g(k) \text{ (see [1], [2], [3], [5]).} \quad (3.1)\]

It will be more convenient to write (3.1) as the equivalent integral form
\[\int_{\mathbb{Z}_p} g(x + n)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = 2\sum_{k=0}^{n-1}(-1)^{n-1-k}g(k). \quad (3.2)\]

Substituting \(g(x) = e^{2xt}\) into the above, we have
\[\int_{\mathbb{Z}_p} e^{(2x+2n)t}d\mu_{-1}(x) + \int_{\mathbb{Z}_p} e^{2xt}d\mu_{-1}(x) = 2\sum_{j=0}^{n-1}(-1)^je^{(2j)t}. \quad (3.3)\]
After some elementary calculations, we obtain

\[
\int_{\mathbb{Z}_p} e^{2xt} d\mu_{-1}(x) = \frac{2}{e^{2t}+1}, \quad \int_{\mathbb{Z}_p} e^{(2x+2n)t} d\mu_{-1}(x) = e^{2nt} \frac{2}{e^{2t}+1}. \tag{3.4}
\]

By using (3.3) and (3.4), we have

\[
\int_{\mathbb{Z}_p} e^{(2x+2n)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} e^{2xt} d\mu_{-1}(x) = \frac{2(1 + e^{2nt})}{e^{2t}+1}.
\]

From the above, we get

\[
\int_{\mathbb{Z}_p} e^{(2x+2n)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} e^{2xt} d\mu_{-1}(x) = \frac{2 \int_{\mathbb{Z}_p} e^{2xt} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} e^{2nt} d\mu_{-1}(x)}. \tag{3.5}
\]

By substituting Taylor series of \(e^{2xt}\) into (3.3), we obtain

\[
\sum_{m=0}^{\infty} \left( \int_{\mathbb{Z}_p} \frac{(2x + 2n)^m + (2x)^m}{m!} d\mu_{-1}(x) \right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left( \sum_{j=0}^{n-1} (-1)^j (2j)^m \right) \frac{t^m}{m!}.
\]

By comparing coefficients \(\frac{t^m}{m!}\) in the above equation, we obtain

\[
\sum_{k=0}^{m} \binom{m}{k} (2n)^{m-k} \int_{\mathbb{Z}_p} (2x)^k d\mu_{-1}(x) + \int_{\mathbb{Z}_p} (2x)^m d\mu_{-1}(x) = 2 \sum_{j=0}^{n-1} (-1)^j (2j)^m
\]

By using (2.1), we have

\[
\sum_{k=0}^{m} \binom{m}{k} (2n)^{m-k} \int_{\mathbb{Z}_p} (2x)^k d\mu_{-1}(x) + \int_{\mathbb{Z}_p} (2x)^m d\mu_{-1}(x) = 2T_m(n-1). \tag{3.6}
\]

By using (3.5) and (3.6), we arrive at the following theorem:

**Theorem 3.1** Let \(n\) be odd positive integer. Then we obtain

\[
\frac{2 \int_{\mathbb{Z}_p} e^{2xt} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} e^{2nt} d\mu_{-1}(x)} = \sum_{m=0}^{\infty} \left( 2T_m(n-1) \right) \frac{t^m}{m!}. \tag{3.7}
\]

Let \(w_1\) and \(w_2\) be odd positive integers. By using (3.7), we have

\[
\frac{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} e^{(w_1 x_1 + w_2 x_2 + w_1 w_2 x_1 x_2)t} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{2w_1 w_2 t} d\mu_{-1}(x)} = \frac{2e^{w_1 w_2 t}(e^{2w_1 w_2 t} + 1)}{(e^{2w_1 t} + 1)(e^{2w_2 t} + 1)}. \tag{3.8}
\]
By using (3.7) and (3.8), after elementary calculations, we obtain
\[
a = \left( \frac{1}{2} \int_{\mathbb{Z}_p} e^{(w_1^2 x_1 + w_2 x_2)^t} d\mu_{-1}(x_1) \right) \left( \frac{2 \int_{\mathbb{Z}_p} e^{2x_1 x_2^t} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{2w_1 x_2} d\mu_{-1}(x)} \right) = \left( \frac{1}{2} \sum_{m=0}^{\infty} T_m(w_2 x) w_1^m \frac{t^m}{m!} \right) \left( 2 \sum_{m=0}^{\infty} T_m(w_1 - 1) w_2^m \frac{t^m}{m!} \right).
\] (3.9)

By using Cauchy product in the above, we have
\[
a = \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \binom{m}{j} T_j(w_2 x) w_1^j T_{m-j}(w_1 - 1) w_2^{m-j} \right) \frac{t^m}{m!}.
\] (3.10)

By using the symmetry in (3.9), we have
\[
a = \left( \frac{1}{2} \int_{\mathbb{Z}_p} e^{(w_2^2 x_2 + w_1 x_2)^t} d\mu_{-1}(x_2) \right) \left( \frac{2 \int_{\mathbb{Z}_p} e^{2x_1 x_2^t} d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p} e^{2w_2 x_2} d\mu_{-1}(x)} \right) = \left( \frac{1}{2} \sum_{m=0}^{\infty} T_m(w_1 x) w_2^m \frac{t^m}{m!} \right) \left( 2 \sum_{m=0}^{\infty} T_m(w_2 - 1) w_1^m \frac{t^m}{m!} \right).
\]

Thus we have
\[
a = \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \binom{m}{j} T_j(w_1 x) w_2^j T_{m-j}(w_2 - 1) w_1^{m-j} \right) \frac{t^m}{m!}.
\] (3.11)

By comparing coefficients \( \frac{t^m}{m!} \) in the both sides of (3.10) and (3.11), we arrive at the following theorem:

**Theorem 3.2** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain
\[
\sum_{j=0}^{m} \binom{m}{j} T_j(w_1 x) T_{m-j}(w_2 - 1) w_1^{m-j} w_2^{j} = \sum_{j=0}^{m} \binom{m}{j} T_j(w_2 x) T_{m-j}(w_1 - 1) w_1^{j} w_2^{m-j},
\]
where \( T_k(x) \) and \( T_m(k) \) denote the tangent polynomials and the alternating sums of powers of consecutive even integers, respectively.

By using Theorem 3.2, we have the following corollary:

**Corollary 3.3** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain
\[
\sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{m-k} w_2^{j-k} T_k T_{m-j}(w_2 - 1)
= \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{j} w_2^{m-j-k} T_k T_{m-j}(w_1 - 1).
\]
By using (3.8), we have
\[ a = \left( \frac{1}{2} e^{w_1 x} \int_{\mathbb{Z}_p} e^{2x_1 w_1 t} d\mu_{-1}(x_1) \right) \left( \frac{2 \int_{\mathbb{Z}_p} e^{2x_2 w_2 t} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{2w_1 x w_2 t} d\mu_{-1}(x)} \right) \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{w_1-1} (-1)^j T_n \left( w_2 x + \frac{2j w_2}{w_1} \right) \right) \frac{t^n}{n!}. \]  
(3.12)

By using the symmetry property in (3.12), we also have
\[ a = \left( \frac{1}{2} e^{w_2 x} \int_{\mathbb{Z}_p} e^{2x_2 w_2 t} d\mu_{-1}(x_2) \right) \left( \frac{2 \int_{\mathbb{Z}_p} e^{2x_1 w_1 t} d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p} e^{2w_1 x w_2 t} d\mu_{-1}(x)} \right) \]
\[ = \sum_{n=0}^{w_2-1} (-1)^j \int_{\mathbb{Z}_p} e^{2x_2 x + \frac{2j w_1}{w_2} (w_2 t)} d\mu_{-1}(x_1) \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{w_2-1} (-1)^j T_n \left( w_1 x + \frac{2j w_1}{w_2} \right) \right) \frac{t^n}{n!}. \]  
(3.13)

By comparing coefficients \( \frac{t^n}{n!} \) in the both sides of (3.12) and (3.13), we have the following theorem.

**Theorem 3.4** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain
\[ \sum_{j=0}^{w_1-1} (-1)^j T_n \left( w_2 x + \frac{2j w_2}{w_1} \right) w_1^n = \sum_{j=0}^{w_2-1} (-1)^j T_n \left( w_1 x + \frac{2j w_1}{w_2} \right) w_2^n. \]  
(3.14)

Substituting \( w_1 = 1 \) into (3.14), we arrive at the following corollary.

**Corollary 3.5** Let \( w_2 \) be odd positive integer. Then we obtain
\[ T_n(x) = w_2^n \sum_{j=0}^{w_2-1} (-1)^j T_n \left( \frac{x + 2j}{w_2} \right). \]

References


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